Lecture 9: Math 285 (Bronski)

Numerical Methods:

Since we will be using the IODE package to study ordinary differential equations we should know at least a little bit about how to numerically simulate such equations. The easiest method is something called an Euler scheme (or Euler Method). An Euler method is as follows: suppose that one wants to simulate

\[ y' = f(y, t) \quad y(0) = y_0 \]

then one “solves” the above equation by integrating up over a short interval \( \Delta t \)

\[ y(\Delta t) - y(0) = \int_0^{\Delta t} f(y(s), s) ds \]

This is a correct equation, of course, but it is not terribly useful since one doesn’t know \( y(s) \). But since we are only integrating over a short interval one can make the approximation that \( y(s) \approx y(0) \) as well as \( s \approx 0 \). This gives the equality

\[ y(\Delta t) = y(0) + \Delta t f(y_0, 0) \]

This gets us one step. To take the next step we make the same approximation

\[ y(2\Delta t) = y(\Delta t) + \Delta t f(y(\Delta t), \Delta t) \]

Let’s denote \( y_i = y(i\Delta t) \). Then the rule for generating the next term becomes

\[ y_{i+1} = y_i + \Delta t f(y_i, i\Delta t) \]

We can, at least approximately analyze the error we make in using this scheme. We are saying that the value of the function \( y(s) \) over the interval \([i\Delta t, (i+1)\Delta t]\) is \( y_i = y(i\Delta t) \). A more accurate approximation is \( y_i = y(i\Delta t) + (s - i\Delta t)f'(i\Delta t) \). Plugging this into the above integral formula gives

\[ y_{i+1} = y_i + \Delta t f(y_i, i\Delta t) + O(\Delta t^2) \]

Now if each step is of size \( \Delta t \) then the total number of steps required to integrate up to time 1 is of the order of \( N = 1/\Delta t \). Thus the total error is of the order of \( \Delta t \).

This is basically the analog for ordinary differential equations of the rectangle rule for numerical evaluation of integrals.

A more accurate method would be a “midpoint” method. Of course the midpoint method would be

\[ y(\Delta t) = y(0) + \Delta t f(\frac{\Delta t}{2}, y(\Delta t/2)) \]

Well, of course we don’t know \( y(\Delta t/2) \) but we can approximate if by \( y(0) + \frac{\Delta t}{2} f(y_0, 0) \). So the first step would be

\[ y(\Delta t) = y(0) + \Delta t f(\frac{\Delta t}{2}, y_0 + \frac{\Delta t}{2} f(y_0, 0)) \]
and successive steps would be given by

\[ y_{i+1} = y_i + \Delta t f\left(\frac{\Delta t}{2}, y_i + \frac{\Delta t}{2} f(y_i, i\Delta t)\right). \]

This is only the tip of the iceberg. Just as there are higher order methods for numerically evaluating integrals there are analogous methods for numerically integrating ordinary differential equations. The most common methods are the Runge-Kutta methods, which come in all sorts of orders. There is a second order Runge-Kutta (The order refers to the size of the Total Error - the Euler scheme is first order, the midpoint scheme and Runge-Kutta-2 are second order schemes - the error incurred in integrating up to time one is of order \(\Delta t^2\). The most common method, which is used in the IODE program, is the fourth order Runge-Kutta method - the total error is \(O(\Delta t^4)\) (meaning that the “local” error is \(O(\Delta t^5)\). In one of the IODE labs you’ll be able to play around with changing the method of integration and see how the various methods behave.

**Example 1.** Consider the ODE

\[ y' = xy - x^4/4 + 1 \quad y(0) = 0 \]

The solution is given by

\[ y = x^3/3 + x \]

The first order Euler scheme applied to this system gives the following graph:
The midpoint scheme gives

Here the step size $\delta t = 0.1$. You can see that the midpoint scheme gives a better result than the Euler scheme. (The RK4 solution would be basically indistinguishable from the true curve).

**Linear Equations of Higher Order**

Recall that a linear equation is one in which $y$ and all of its derivatives enter in a linear fashion. The general linear equation is

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Again the homogeneous linear equation is

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Some examples of equations which are not linear are

$$(1) \quad y'' + yy' = 0$$

$$(2) \quad y' = (1 + y^2)$$

$$(3) \quad y''' + \sin(y) = y'$$

In the first case we have a $yy'$ term which is not linear. In the second we have a $y^2$ term, and the third we have a $\sin(y)$ term.

The following are some examples of linear equations

$$(4) \quad y' = y$$

$$(5) \quad y'' = -y$$

$$(6) \quad y'' + x^2 y = Ey$$

The first main theorem is
**Theorem 1. Superposition Principle:** If \( y_1(x) \) and \( y_2(x) \) are both solutions of a homogeneous linear equation then any linear combination 
\[
a y_1(x) + b y_2(x)
\]
is also a solution.

**Example 2.** Consider the equation of the harmonic oscillator:
\[
y'' = -y
\]
It is clear that \( y(x) = \sin(x) \) and \( y(x) = \cos(x) \) are both solutions of the above equation. From the above theorem we know that
\[
y(x) = a \sin(x) + b \cos(x)
\]
is a solution. Since there are two constants of integration we expect that this is the general solution. We’ll talk more about this concept (general solutions/linear independence) later.

I wanted to mention the second main theorem, the existence and uniqueness for linear equations. This is a special case of a theorem that we have already learned, but we’ll restate it for clarity.

**Motivation: do first order linear case**

**Theorem 2.** Suppose that the functions \( p(x), q(x) \) and \( f(x) \) are continuous in an open interval containing the point \( a \). Then, given any two numbers \( b_1 \) and \( b_2 \) the equation
\[
y'' + p(x)y' + q(x)y = f(x) \quad y(0) = b_1 \quad y'(0) = b_2
\]
**Example 3.** Consider again the equation
\[
y'' = -y
\]
The most general solution is
\[
y = a \sin(x) + b \cos(x)
\]

**Example 4.** Consider the equation
\[
x^2 y'' - 4 xy' + 6 y = 0
\]
This has two solutions \( y = x^2 \) and \( y = x^3 \). Both of these satisfy the initial data \( y(0) = 0, y'(0) = 0 \). But we have a uniqueness theorem. What gives?