Lecture 7: Math 285 (Bronski)
The phase line, equilibria, and stability II - Bifurcations

Last time we were considering ordinary differential equations of the form

\[ y' = f(y) \]

for which we developed a way to understand qualitatively the behavior of the solutions without actually bothering to solve them.

The basic idea was this: zeros of the function \( f(y) \) represent equilibria - if you start there you stay there. We can get an idea of how these solutions behave by calculating all of the equilibria and their stability. Recall that if \( y_0 \) is an equilibrium then it is stable (meaning nearby solutions are attracted to \( y_0 \)) if \( f'(y_0) < 0 \) and it is unstable if \( f'(y_0) > 0 \). For instance for the equation

\[ \frac{dy}{dt} = y(1 - y)(2 + y) \]

I want to talk a little bit about something called bifurcation theory, which is closely connected with the phase line analysis. A good example is given by the logistic model with harvesting.

\[ \frac{dy}{dt} = ky(P_0 - y) - h \]

Again this is supposed to represent the growth of a population that grows according to the logistic equation but has population removed at a constant rate (fishing, hunting, etc.)

Recall that the behavior depended on the harvesting rate \( h \). If \( h \) is small enough, specifically if \( P_0^2 > 4h/k \), then there are two equilibria. The lower population is unstable, and the upper population is stable. The picture looks
There is a qualitative change in behavior as $h$ increases. As $h$ grows the upper equilibrium moves down and the lower one moves up. At the critical value $h = 1/4P_0^2k$ the two equilibria merge, and give a “semistable” equilibrium. When $h$ exceeds this critical value there are no equilibria, and the population crashes.

It is often useful to encapsulate this information into a bifurcation diagram. A bifurcation diagram has the equilibria (usually together with stability) drawn as a function of the bifurcation parameter (here $h$.). For instance the bifurcation diagram for the logistic model with harvesting looks like

There are lots of types of bifurcation, and many of them have names\textsuperscript{1}. One

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\textsuperscript{1}My favorite is named “Snake”
of the most common is called the pitchfork bifurcation. An equation which exhibits a pitchfork bifurcation is

\[
\frac{dy}{dt} = y - ay^3
\]

The roots of this are \( y = 0 \) (for any value of \( a \)) and \( y = \pm \frac{1}{\sqrt[3]{a}} \) (for \( a > 0 \)). What is the stability of each?

Another example is given by the following equation

\[
\frac{dy}{dt} = -y^3 + y + h
\]

There is, as I have remarked before, a formula for the roots of the cubic but it is effectively useless. Fortunately it isn’t too hard to figure out how many roots there are for various values of \( h \), and what the stability properties of each of
Bifurcation theory is actually extremely useful in engineering, as many devices are designed using bifurcation theory and similar ideas. Any guesses what kind of device has the bifurcation diagram above?
Numerical Methods

I want to talk a bit about some numerical methods for solving differential equations. The simplest is the Euler method. The basic idea of the Euler method (and of almost every other numerical method) is to replace a differential equation by a difference equation: basically a sum. The standard Euler scheme is this: If one wants to solve the equation

\[ \frac{dy}{dt} = f(y, t) \quad y(0) = y_0 \]

then think about taking a short step of length \( \Delta t \). The above equation can be integrated up to give

\[ \int_0^{\Delta t} \frac{dy}{dt} dt = \int_0^{\Delta t} f(y(t), t) \]

Now normally this would not be useful, but we are going to take advantage of the fact that \( \Delta t \) is small. The above is approximately given by

\[ y(\Delta t) - y(0) \approx f(y(0), 0) \Delta t \]

What is there error in making this approximation? It is of the order \( \frac{\partial f}{\partial y} (\Delta t)^2 \). Now we are only going a short distance \( \Delta t \), so to really get somewhere we need to take a lot of little steps: to get out to time \( t = 1 \) we need to take of the order of \( (\Delta t)^{-1} \) steps. But this is still a win, since the error is small:

Of course, there are better things to do than the Euler scheme. The above is basically like the rectangle rule for doing integrals. There are much better schemes that one can implement. For instance