Existence, Uniqueness, and Independence:

Last time we began talking about the existence and uniqueness of solutions to differential equations. Let me remind you of the two theorems we stated:

The first is the existence theorem for linear equations

**Theorem 1 (Existence and Uniqueness).** Consider the second order linear differential equation

\[ y'' + p(x)y' + q(x)y = f(x) \quad y(a) = b_1 \quad y'(a) = b_2. \]

If \( p(x), q(x), f(x) \) are all continuous in a neighborhood of \( a \) then the above equation has a unique solution.

The second is the principle of superposition:

**Theorem 2 (Superposition Principle).** Consider the second order linear homogeneous differential equation

\[ y'' + p(x)y' + q(x)y = 0 \]

If \( y_1(x) \) and \( y_2(x) \) are both solutions to the above equation then an arbitrary linear combination, \( Y(x) = ay_1(x) + by_2(x) \) is also a solution.

One example we considered was that of the second order equation

\[ y'' + y = 0 \]

It was not hard to guess the two solutions \( \sin(x), \cos(x) \). By the superposition theorem \( y = a\sin(x) + b\cos(x) \) is also a solution. If one is given the equation with boundary conditions

\[ y'' + y = 0 \quad y(0) = b_1 \quad y'(0) = b_2 \]

it is not hard to see that one solution of the above is \( y = b_1 \cos(x) + b_2 \sin(x) \) - as shown last time we just assume a linear combination of \( \sin(x), \cos(x) \) and solve for the coefficients. By the first theorem this must be the ONLY solution, since solutions are unique.

The same thing holds in general. We’ll state this as a “guiding principle”.

**Guiding Principle:** If we can find two different solutions to a second order linear homogeneous differential equation, then the general solution will be given by a linear combination of these two solutions.

The important thing here is “different.” In the above example we could have taken a linear combination of \( \sin(x) \) and \( 2\sin(x) \). By the superposition principle this would have solved the differential equation. However this would not have been the most general solution. For instance the solution to

\[ y'' + y = 0 \quad y(0) = 1 \quad y'(0) = 0 \]
is \( y = \cos(x) \), which is \textit{not} a linear combination of \( \sin(x) \) and \( 2\sin(x) \).

So the important thing is to distinguish when two solutions are really “different”. This is not always so clear, especially in the case of higher order equations. We’ll start with a simple definition, which will need to be modified slightly in the case of higher order equations.

**Linear Independence:** Two functions \( f(x) \) and \( g(x) \) defined on some open interval are said to be linearly independent if neither one is a constant multiple of the other. Otherwise they are linearly dependent.

**Example 1.** The functions \( \sin(x) \) and \( 2\sin(x) \) are linearly dependent since each is a multiple of the other.

**Example 2.** The functions \( \sin(x) \) and \( \cos(x) \) are linearly independent: neither one is a multiple of the other. To see this suppose that we had

\[
\sin(x) = c\cos(x)
\]

Plugging in \( x = 0 \) gives \( c = 0 \). Plugging in \( x = \frac{\pi}{4} \) gives

\[
\frac{\sqrt{2}}{2} = c\frac{\sqrt{2}}{2}
\]

or \( c = 1 \). So there is no single choice of \( c \) that will work.

**Example 3.** The functions \( e^x \) and \( e^{-x} \) are linearly independent. (SHOW!)

One of the main tools for deciding the linear independence of a pair of solutions is the Wronskian, which is a determinant built out of a pair of solutions. The Wronskian “detects” when a pair of solutions is linearly dependent.

**Definition:** Given two functions \( f(x) \), \( g(x) \) the Wronskian is defined to be

\[
W(f, g) = \left| \begin{array}{cc}
  f(x) & g(x) \\
  f'(x) & g'(x)
\end{array} \right| = f(x)g'(x) - f'(x)g(x)
\]
The first thing to notice is that this thing is zero when $f$ is a constant multiple of $g$.

**Example 4.** Some examples: The Wronskian of the two solutions of

$$y'' + y = 0$$

is

Similarly the Wronskian of the functions $e^x, e^{-x}$, which both satisfy the equation

$$y'' - y = 0$$
The next thing to notice is that the Wronskian actually satisfies a first order equation, so we can explicitly write down the solution:

**Theorem:** Suppose that \( y_1 \) and \( y_2 \) are two solutions to the equation

\[
A(x)y'' + B(x)y' + C(x)y = 0
\]

Then the Wronskian \( W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) \) solves the FIRST ORDER differential equation

\[
A(x)W' + B(x)W = 0
\]

**Proof**

**Example 5.** *Since \( \sin(x) \) and \( \cos(x) \) are both solutions of*

\[
y'' + y = 0
\]

- *in other words \( p(x) = 0 \) and \( q(x) = 1 \) it follows that the Wronskian solves*

\[
W' = 0
\]
in other words the Wronskian is a constant independent of $x$. We already know this to be the case.

**Example 6.** We saw last time that two solutions to

$$x^2 y'' - 4xy' + 6y = 0$$

are $y_1 = x^2$ and $y_2 = x^3$. It follows that the Wronskian solves

$$x^2 W' - 4x W = 0$$

which has the solution $W = cx^4$. Computing the Wronskian of $y_1$ and $y_2$ gives

Notice that, since the Wronskian is given by

$$W(x) = W(0)e^{-\int p(x)dx}$$

we have the following result:

**Theorem:** Suppose that $y_1$ and $y_2$ are two solutions to the equation

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous in some interval $I$. Then the Wronskian $W(x)$ is either identically 0 in $I$ or it is **NEVER** zero in $I$. 
proof: