Math 285 Homework # 4
Due Friday Sept. 18 in class.

Section 2.2 # 19,21,29

Section 2.3 # 1,4,6

2.2.19 Consider the model for a population with harvesting given by

\[
\frac{dx}{dt} = f(x) = \frac{1}{10}x(10 - x) - h
\]

where \( h \) is the harvesting rate and 10 represents the equilibrium population (in kilo-makerels) in the absence of harvesting.

First note that the equilibria are given by the roots of the quadratic

\[-10f(x) = x^2 - 10x + 10h = 0\]

which has the roots

\[x = \frac{10 \pm \sqrt{100 - 40h}}{2} = 5 \pm 5\sqrt{1 - \frac{2h}{5}}\]

There are two real roots as long as \( h < \frac{5}{2} \), a double root for \( h = \frac{5}{2} \) and no real roots for \( h > \frac{5}{2} \). The double root occurs at \( x = 5 \) when \( h = \frac{5}{2} \).

The stability is determined by the sign of \( f' \) at the root. It is easy to see that

\[f'(x) = 1 - \frac{x}{5}\]

so the derivative is positive for \( x < 5 \) and negative for \( x > 5 \) and vanishes at \( x = 5 \) (The derivative will always vanish at the bifurcation.)
This implies that the upper branch (larger population) is stable and the lower branch is unstable.

2.2.21

\[ \frac{dx}{dt} = f(x) = kx - x^3 \]

This is the basic example of something known as the pitchfork bifurcation. The equilibrium solutions of the above equation are

\[ f(x) = kx - x^3 = x(k - x^2) = 0 \]

the roots of which are given by

\[ x = 0 \text{ all } k \]
\[ x = \pm \sqrt{k}, \quad k > 0 \]
It is easy to see that \( f'(0) = k \). Thus \( x = 0 \) is stable for \( k < 0 \) and unstable for \( k > 0 \). It is also easy to see that \( f'(\pm \sqrt{k}) = -2k \) and thus these branches are stable. So the bifurcation diagram looks like this

\[
\frac{dx}{dt} = f(x) = (x - a)(x - b)(x - c)
\]

There are two ways to do this. Note that the equilibria are \( x = a, x = b, x = c \). Evaluating the derivative

\[
f'(x) = (x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)
f'(a) = (a - b)(a - c) > 0
f'(b) = (b - a)(b - c) < 0
f'(c) = (c - a)(c - b) > 0
\]

so the equilibria \( a, b, c \) are unstable, stable, unstable respectively. Alternatively one can imagine taking a point between \( a, b \). In this region \( x - a > 0, x - b < 0 \) and \( x - c < 0 \) so \( f' \) is positive. Similarly if \( x \) is
between $b, c$ then $x - a > 0$, $x - b > 0$ and $x - c < 0$ so $f'$ is negative. Filling in the reset of the arrows gives the following phase line:

For the equation

$$\frac{dx}{dt} = f(x) = (a - x)(x - b)(x - c)$$

notice that the righthand side is just the negative of the righthand side in the first problem, so all of the signs switch

2.3.1
The first paragraph translates to the following equation:
\[
\frac{dv}{dt} = k(250 - v)
\]

This can be solved as first order linear or as separable. To do the latter note that

\[
\int \frac{dv}{250 - v} = k \int dt
\]

\[
-\ln|250 - v| = kt + c
\]

\[
250 - v = e^{-(kt+c)} = ae^{-kt}
\]

\[
v = 250 - ae^{-kt}
\]

We are further told that the car starts from rest, so \(v(0) = 0\) and \(a = 250\). We’re also told that \(v(10) = 100\). Thus

\[
100 = 250 - 250e^{-10k}
\]

\[
250e^{-10k} = 150
\]

\[
k = -\frac{1}{10} \ln\left(\frac{150}{250}\right) = \frac{1}{10} \ln\left(\frac{250}{150}\right) \approx .051
\]

Finally we’re asked how long to accelerate to \(200\, \text{km/hr}\). Then we have

\[
200 = 250 - 250e^{-0.051t}
\]

\[
250e^{-0.051t} = 50
\]

\[
k = -\frac{1}{.051} \ln\left(\frac{1}{5}\right) \approx 31.5 \, \text{s}
\]

2.3.4

\[
\frac{dv}{dt} = -kv^2
\]

\[-\int_{v_0}^{v(t)} \frac{dv}{v^2} = \int_0^t k dt\]

\[
\frac{1}{v(t)} - \frac{1}{v_0} = kt
\]

\[
v(t) = \frac{1}{kt + \frac{1}{v_0}} = \frac{v_0}{1 + v_0kt}
\]
Since \( v(t) = \frac{dx}{dt} \), this can in turn be integrated up to give

\[
\frac{dx}{dt} = \frac{v_0}{1 + v_0 kt}
\]

\[
\int_0^t \frac{dx}{dt} dt = \int_0^t \frac{v_0}{1 + v_0 kt} dt
\]

\[
x(t) - x(0) = \frac{1}{k} \ln |1 + v_0 kt|
\]

Note that the total distance travelled goes to infinity as \( t \to \infty \) since the log function grows without bound. This is in contrast to the damping which is proportional to \( v \) (problem 2) where the total distance traveled is finite. The way to think about this is as follows: when the body is traveling very slowly \( v^2 \) is going to be much smaller than \( v \). This means that the \( v^2 \) drag is much less effective at slowing down a slowly moving body. This is why this drag law allows the body to travel infinitely far, while the linear drag law, which is more effective at slowing down slowly moving bodies, only allows a finite travel distance.

2.3.6

\[
\frac{dv}{dt} = -kv^{\frac{3}{2}}
\]

\[- \int_{v_0}^{v(t)} \frac{dv}{v^{\frac{3}{2}}} = \int_0^t k dt\]

\[2 \left( \frac{1}{v^{\frac{1}{2}}(t)} - \frac{1}{v_0^{\frac{1}{2}}} \right) = kt\]

\[
v(t) = \frac{1}{(v_0^{-\frac{1}{2}} + \frac{kt}{2})^2}
\]

\[= \frac{v_0}{(1 + \frac{k v_0^{\frac{3}{2}} t}{2})^2}\]

This integral is easy to do, but you don’t really need to so. Just notice that \( v(t) \) falls off like \( t^{-2} \) This means that the integral of \( v \) over all time will be finite. This means that given infinite time the body will only travel finitely far. Again this is because a \( v^\frac{3}{2} \) power drag is more effective at low velocities than a linear drag law.
If you want to actually do the integral you’ll find the following:

\[
\frac{dx}{dt} = \frac{v_0}{(1 + \frac{k\sqrt{v_0}t}{2})^2}
\]

\[
\int_0^t \frac{dx}{dt} dt = \int_0^t \frac{v_0}{(1 + \frac{k\sqrt{v_0}t}{2})^2}
\]

\[
x(t) - x(0) = \frac{2\sqrt{v_0}}{k} \left(1 - \frac{1}{1 + \frac{k\sqrt{v_0}t}{2}}\right)
\]

As in the linear damping we see that the body travels only a finite distance.