Practice Midterm Exam 1: Math 285 (Bronsiki)

Instructions: Answer each question clearly and completely, showing all necessary work. If you are using a theorem STATE the theorem as carefully as possible. Answers with insufficient explanation will not receive credit, regardless of correctness.

Name:
2 Problem 1: 18pts 1a: State the existence and uniqueness theorem for first order differential equations

\[ y' = f(y, x) \ y(a) = b \]

Solution 1 (1a). The first order differential equation

\[ y' = f(y, x) \ y(a) = b \]

has a solution as long as \( f(y, x) \) is continuous in a neighborhood of \( y = b, x = a \). This solution is unique as long as \( \frac{\partial f}{\partial y} \) is also continuous in a neighborhood of \( y = b, x = a \).

1b: State the existence and uniqueness theorem in the special case of a first order linear equation

\[ y' + p(x)y = q(x) \ y(a) = b \]

Solution 2 (1b). The solution exists and is unique as long as \( p(x) \) and \( q(x) \) are continuous.

c: Is the ordinary differential equation

\[ y' = |y|^{\frac{3}{2}} \ y(0) = 0 \]

guaranteed to have a unique solution. Explain your reasoning completely.

Solution 3 (1c). YES: The functions

\[ f(y) = |y|^{\frac{3}{2}} \]

and

\[ \frac{\partial f}{\partial y} = \frac{3}{2} |y|^{\frac{1}{2}} \text{sign}(y) \]

are both continuous near \( y = 0 \).
Problem 2 [16pts] a: Give the formula for the integrating factor for the first order linear equation

\[ y' + p(x)y = q(x) \]

Solution 4 (2a). The integrating factor is given by \( \mu(x) = e^{\int p(x)dx} \)

b: Solve the first order linear equation

\[ y' + \left( \frac{1}{x} - \cos(x) \right)y = e^{\sin(x)} \quad y(1) = 0. \]

Solution 5 (2b). The integrating factor is given by

\[ \mu(x) = e^{\int p(x)dx} = e^{\ln(x) - \sin(x)} = xe^{-\sin(x)} \]

Integrating through by this gives

\[ \frac{d}{dx}(yxe^{-\sin(x)}) = x \quad y(1) = 0 \]

\[ yxe^{-\sin(x)} = \frac{x^2}{2} + c \]

\[ y = \frac{x}{2}e^{\sin(x)} + ce^{\sin(x)}/x \]

Solving for the constant \( c \) gives \( c = -1/2 \) and

\[ y = \frac{x}{2}e^{\sin(x)} - \frac{e^{\sin(x)}}{2x} \]
Problem 3 [16pts] a: Find the solution to

\[ y' = x(1 - y) \quad y(0) = 5 \]

Find \( y \) explicitly as a function of \( x \).

Solution 6 (3a). Using the fact that this is separable we get

\[
\int \frac{dy}{1 - y} = \int x \, dx
\]

\[-\ln|1 - y| = x^2 + c
\]

\[ 1 - y = Ce^{-x^2} \]

\[ y = 1 - Ce^{-x^2} \]

where the constants \( C \) and \( c \) are related by \( C = e^c \).

b: What is the behavior of \( y(x) \) in the limit as \( x \to +\infty \).

Solution 7 (3b). As \( x \to \infty \) we see that \( y \to 1 \)
Problem 4 [18pts]: Classify the following equations according to

- Order
- Linear/Nonlinear
- If the equation is linear, classify it as homogeneous or inhomogeneous.

(1) \((x^{15} + e^x)y'' + (\cos(x)y' - e^{x^3}y) = 11\)

(2) **LINEAR second order non-homogeneous**

(3) \(\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2}y + \frac{dy}{dx} = 8\)

(4) **NONLINEAR fourth order**

(5) \(y'' + \sin(x - y) = 0\)

(6) **NONLINEAR second order**

(7) \(x^2y'' + xy' + (x^2 - 8)y = 0\)

(8) **LINEAR second order homogeneous**

(9) \(\frac{d^3y}{dx^3} + \sin\left(\frac{d^2y}{dx^2}\right) = 0\)

(10) **Nonlinear third order**

(11) \(\frac{d^{2008}y}{dx^{2008}} + 11y = y^3\)

(12) **Nonlinear 2008 th order**
Problem 5 [16pts]: Draw the slope field for the equation

\[ y' = (x - y) \]

in the space provided, along with some sample solutions curves.
Problem 6 [16pts]: Find the solution to the following differential equation

\[ y'' + 5y' + 4y = 0 \quad y(0) = 2 \quad y'(0) = 1 \]

Solution 8 (6). The characteristic polynomial is given by

\[ r^2 + 5r + 4 = (r + 1)(r + 4) = 0 \]

The roots of this are \( r = -1, r = -4 \). Thus the general solution is given by

\[ y = Ae^{-x} + Be^{-4x} \]

Solving for the constants we see that

\[ y(x) = Ae^{-x} + Be^{-4x} \]
\[ y(0) = A + B = 2 \]
\[ y'(x) = -Ae^{-x} - 4Be^{-4x} \]

\[ y'(0) = -A - 4B = 1 \]

which gives \(-3B = 3\) or \( B = -1 \) and \( A = 3 \) thus

\[ y(x) = 3e^{-x} - e^{-4x} \]