9.2 Parametric Curves II - Calculus

Last time we began discussing how to do calculus on a curve defined parametrically. The first thing we discussed was the tangent vector, which is the natural extension of the derivative of a function. Recall that

Definition: Given a parametric curve defined by \((x(t), y(t))\) the tangent vector (velocity vector) is given by \((\frac{dx}{dt}, \frac{dy}{dt})\).

Again one example we gave last time was the cycloid curve

\[ x(t) = t - \sin(t) \quad y(t) = 1 - \cos(t) \]

The tangent vector is given by

\[ \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1 - \cos(t), \sin(t)) \]

Again we note that when \(t = 0, 2\pi, 4\pi\) the velocity vector is \((0, 0)\), so the curve is instantaneously at rest. This makes sense - these are the points where the curve turns around. At \(t = \pi, 3\pi, 5\pi\) the velocity vector is given by \((\frac{dx}{dt}, \frac{dy}{dt}) = (1 - \cos(t), 0)\). Again this makes sense physically - the motion is all in the \(x\) direction with no component in the \(y\) direction. This makes sense since the curve has a maximum in the vertical direction at these points.

It is easy to relate this to the usual derivative \(\frac{dy}{dx}\). By the chain rule we have that

\[ \frac{dy}{dt} = \frac{dx}{dt} \frac{dy}{dx} \]

which we can solve for \(\frac{dy}{dx}\) to give

\[ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \]

So in terms of the ordinary derivative we have \(\frac{dy}{dx} = 0\) at \(t = \pi, 3\pi, 5\pi, \ldots\) (which corresponds to \(x = \pi, 3\pi, 5\pi, \ldots\)) Again this makes sense because the function has a maximum there. The derivative goes to infinity as \(x \to 0, 2\pi, 4\pi, \ldots\). This makes sense because (as the picture makes clear) the slope is infinite there.

Example:

Another example is given in the text: The carnival ride called the Scrambler (see Ex 2.1 Section 9.2) has the equation

\[ x(t) = 2 \cos(t) + \sin(2t) \quad y(t) = 2 \sin(t) + \cos(2t) \]

The tangent vector to the curve has the form

\[ \frac{dx}{dt} = 2 \cos(2t) - 2 \sin(t) \quad \frac{dy}{dt} = 2 \cos(t) - 2 \sin(2t) \]
Figure 1: The cycloid curve with derivatives and tangent vectors.
Let’s compute the speed at which a point on the curve is moving - the length of the velocity vector. It is easy to see that

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = (2 \cos(2t) - 2 \sin(t))^2 + (2 \cos(t) - 2 \sin(2t))^2 \]

\[ = 4 \cos^2(2t) - 8 \cos(2t) \sin(t) + 4 \sin^2(t) + 4 \cos^2(t) - 8 \cos(t) \sin(2t) + 4 \sin^2(2t) \]

\[ = 8 - 8 \cos(2t) \sin(t) - 8 \cos(t) \sin(2t) \]

\[ = 8 - 8 \sin(3t) \]

where the last step follows from the identity \( \sin(a+b) = \sin(a) \cos(b) + \sin(b) \cos(a) \).

Thus we can see that the speed is zero when \( \sin(3t) = 1 \) which occurs when

\[ 3t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \]

The speed is largest \( \sqrt{16} = 4 \) when \( \sin(3t) = -1 \) which occurs when

\[ t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \ldots \]

**Area:**

It is also useful to be able to find the area enclosed by a (closed) parametric curve. Suppose the parametric equation

\[ (x(t), y(t)) \quad t \in [a, b) \]

defines a closed curve. The the area enclosed is given by

\[ \text{Area} = \int_a^b y(t) \frac{dx}{dt} dt = - \int_a^b x(t) \frac{dy}{dt} dt \]

There are a couple of reasonably straightforward ways to see this. The first is
to relate this to the usual integral:

The second is to draw the following picture

And note that

\[ \theta = \arctan\left(\frac{y}{x}\right) \]

and thus

\[ \frac{d\theta}{dt} = \frac{\frac{dy}{dt} x^{-1} - y \frac{dx}{dt} x^{-2}}{x^2 + y^2} \]