1 Elementary substitutions/ u-substitutions

I want to begin with a brief review of u-substitutions, which should be familiar from Calculus I, and then proceed to integration by parts.

The basic idea of the u-substitutions (or elementary substitution) is to use the chain rule to recognize the integrand as an exact derivative. Lets begin with an easy example

Example 1. *Compute the (definite) integral*

$$I = \int_0^{4\pi} \sin(\frac{x}{4})dx$$

Solution 1. *We begin by making the elementary substitution $u = \frac{x}{4}$. Whenever we do a change of variables we need to compute the way in which the differential changes, and we need to compute how the interval of integration changes. The change in the differential is easy: we have*

$$\begin{aligned} u &= \frac{x}{4} \\ du &= \frac{dx}{4} \\ 4du &= dx \end{aligned}$$

The change in the interval of integration is also easy: since $x \in (0, 4\pi)$ we have the $u = \frac{x}{4} \in (0, \pi)$. Thus we have

$$\begin{aligned} I = \int_0^{4\pi} \sin(\frac{x}{4})dx &= \int_0^{\pi} \underbrace{\sin(u)}_{\sin(\frac{x}{4}}\underbrace{4du}_{dx} \\ &= -4\cos(u)|_0^{\pi} \\ &= -4\cos(\pi) + 4\cos(0) = -4(-1) + 4(1) = 8 \end{aligned}$$

Remember that when making any kind of substitution in an integral $\int_a^b f(x)dx$ one has to transform

- The function itself ($f(x)$).
- The differential dx.
- The limits of integration (assuming a definite integral).

In Calc I you should have learned some other elementary substitutions. Here is one example.

Example 2. *Compute the following indefinite integral*

$$I = \int xe^{-11x^2}dx$$

Solution 2. *The most logical thing to try is to substitute for the thing in the integrand. If we try the substitution*

$$u = -11x^2$$
$$du = -22xdx$$

The the integral becomes

$$I = \underbrace{\int xe^{-11x^2}dx}_{u=-11x^2 \qquad du=-22xdx} = \int \underbrace{e^u}_{e^{-11x^2}} \underbrace{\frac{-1}{22}du}_{xdx}$$
$$= \frac{-1}{22}e^u + c = \frac{-1}{22}e^{-11x^2} + c$$

Note that in this calculation we had to group the factor of x with the dx to get a du. If you had forgotten to transform the differential you would have gotten an extra factor that would have lead to an integral of the form

$$\int u^{\frac{1}{2}} e^u du$$

which you would not have been able to do (it is not expressible in terms of elementary functions - it is something known as the error function, which arises a lot in probability and statistics.

Here is a list of integrals. See if you can guess the correct substitutions:

$$\int \frac{x}{1+x^2}dx$$
$$\int \frac{x^2}{1+x^3}dx$$
$$\int x^5 e^{x^2} dx$$
$$\int \frac{dx}{x \ln^n |x|}$$
$$\int x^2 \sin(14x^3)dx$$
$$\int \sin(x)e^{-2\cos(x)}dx$$

In three of the above cases making the correct substitution leads to an integral that you should be able to do by inspection. In one case (the second integral) the correct substitution leads to an integral that you should be able to do by a technique called integration by parts.

ASIDE: *We are going to learn a number of techniques for calculating integrals in this class. You should remember from Calc I that the procedures of Integration and Differentiation are inverses to one another: The fundamental Theorem tells use that the derivative of the integral is the function itself, and* *the integral of the derivative is the function (plus an arbitrary constant of integration). Most of the techniques for doing integrals correspond to a rule for computing derivatives. For example the method of substitutions corresponds to the chain rule for derivatives. Our next technique, integration by parts, corresponds to the product rule for derivatives.*

2 Integration By Parts

Integration by parts is based on the product rule for derivatives. It is often useful when one has an integral where the integrand can be made to take the form of a product. For instance

$$\int x\sin(x)dx$$

In many cases, as well as all cases where the integrand can be expressed as finite sums and products of

- sin or cos
- Exponentials
- Polynomials

the integral can be done by integration by parts. Basically the idea behind integration by parts is to recognize the integrand as the derivative of a product plus a "correction" term which is simpler to integrate.

To start with lets look at the integral above. We have the identity

$$\frac{d}{dx}(-x\cos(x)) = x\sin(x) - \cos(x)$$

Thus we have

$$\begin{aligned}\int x\sin(x)dx &= \underbrace{\int \frac{d}{dx}(-x\sin(x))dx}_{Fundamental\ Theorem} + \int \cos(x)dx \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x)\end{aligned}$$

Taking the derivative shows that this is, in fact, correct

$$\frac{d}{dx}\left(-x\cos(x) + \sin(x)\right) = x\sin(x) - \cos(x) + \cos(x) = x\sin(x).$$

Now there is a fairly *systematic* way to recognize the integrand as a derivative. The integration by parts formula is usually written in the following way

Fundamental Identity for Integration by Parts

$$\int udv + \int vdu = \int d(uv) = uv$$

or equivalently

$$\int u dv = uv - \int v du$$

To illustrate this lets do a related integral

Example 3. *Compute the integral*

$$\int x \cos(2x) dx$$

SOLUTION: *To do this we use the integration by parts formula:*

$$\begin{aligned}\int \underbrace{x}_{u} \underbrace{\cos(2x)dx}_{dv} &= \int u dv \\ &= uv - \int v du\end{aligned}$$

Now we know that we have

$$\begin{aligned}u(x) &= x \\ dv &= \cos(2x)dx\end{aligned}$$

And we have to figure out v and du respectively. This amounts to integrating one quantity and differentiating the other:

$$\begin{aligned}u &= x \qquad \Longrightarrow \qquad du = dx \\ dv &= \cos(2x)dx \qquad \Longrightarrow \qquad v = \int \cos(2x)dx = \frac{\sin(2x)}{2}\end{aligned}$$

This gives us

$$\begin{aligned}\int \underbrace{x}_{u} \underbrace{\cos(2x)dx}_{dv} &= \int u dv \\ &= uv - \int v du \\ &= x\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2} dx \\ &= x\frac{\sin(2x)}{2} + \int \frac{\cos(2x)}{4} dx\end{aligned}$$

Note that in this example we chose to write the x term as u and the $\cos(x)dx$ term as dv. This is intentional. You can integrate by parts in both directions, but typically only one is going to work. Lets see what happens if we try to work it the other way:

Moral: Integration by parts has a direction. Typically integrating by parts in one direction will work, but the other direction will not. You'll have to develop some experience to know which way is going to work. As a rule of thumb you'll usually try to choose the direction so that things get simpler. For instance if there is a polynomial you will often (THOUGH NOT ALWAYS - SEE EXAMPLES 4,5 BELOW) differentiate it, since this gives a polynomial of lower degree, making things simpler.

Now lets try a few more examples:

$$\int xe^x dx$$

$$\int \frac{x}{\cos^2(x)} dx$$

$$\int x^3 e^{x^2} dx$$

$$\int \ln|x| dx$$

$$\int x^2(\ln|x|) dx \qquad \int \arctan(x) dx$$

One technique which arises a lot is the following: often by making one or more integrations by parts you can relate the original integrand to itself. This lets you algebraically solve for the integrand. Let's start with a simple example. Consider the integral

$$\int x^2 dx = \int x \cdot x dx$$

Now obviously I know from Calc I that the answer is $\frac{x^3}{3} + c$ but lets try doing integration by parts and see what we get.

So integrating by parts lets us relate the original integral to itself and thus compute the integral.

Now this example is a little bit contrived, so lets try it on something less contrived: I want to compute

$$\int e^x \sin(x)dx$$

by the same method. The general idea is the same although you'll see that I have to integrate by parts *twice* to get something of the same form

Here are a few other integrals that can be solved by this sort of calculation

$$\int \sin(ax)\cos(bx)dx$$

$$\int \cos(ax)\cos(bx)dx$$

$$\int e^{ax}\cos(bx)dx$$

Some difficult examples:

$$\int x^2 \arctan(x) dx$$

$$\int x(\ln|x|)^2 dx$$