Quasilinearization, Enflo’s 2-roundness and Aleksandrov’s curvature

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This is a joint work with Professor I.D. Berg.
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We characterize **Aleksandrov $\mathbb{R}_0$ domains**, also known as CAT (0)-spaces, by introducing a *quasilinearization* for an abstract metric space and by employing an analogy between quasilinearization and some characteristic properties of inner product spaces.
Introduction

- This is a joint work with Professor I.D. Berg.
- We characterize *Aleksandrov* $\mathbb{R}_0$ *domains*, also known as CAT (0)-*spaces*, by introducing a *quasilinearization* for an abstract metric space and by employing an analogy between quasilinearization and some characteristic properties of inner product spaces.
- $\mathbb{R}_0$ *domain* is a geodesically connected metric space of non-positive *Aleksandrov’s* curvature, where shortests depend continuously on their ends.
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We characterize Aleksandrov $\mathbb{R}_0$ domains, also known as CAT (0)-spaces, by introducing a quasilinearization for an abstract metric space and by employing an analogy between quasilinearization and some characteristic properties of inner product spaces.

$\mathbb{R}_0$ domain is a geodesically connected metric space of non-positive Aleksandrov’s curvature, where shortests depend continuously on their ends.

Aleksandrov’s definition of curvature $\leq 0$ requires the notion of the upper angle and excess of a geodesic triangle.
Upper angle

Definition

$$\cos \gamma_{LN}(X, Y) = \frac{x^2 + y^2 - z^2}{2xy}.$$ 

Definition

$$\overline{\gamma}_{LN} = \lim_{X, Y \to P} \gamma_{LN}(X, Y).$$
Definition

Excess of the geodesic triangle $\mathcal{T} = ABC$ is defined by

$$\delta (\mathcal{T}) = \bar{\alpha} + \bar{\beta} + \bar{\gamma} - \pi.$$
Aleksandrov’s domain

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  \delta (\mathcal{T}) \leq 0,
  \]
  
  for every geodesic triangle $\mathcal{T}$ in $\mathcal{R}_0$.

- An $\mathcal{R}_0$ domain need not be a manifold and its metric need not be given by a smooth metric tensor.
Examples

Quasilinearization and curvature

\( \mathbb{R}_0 \) domain

Example

Quasilinearization

Quasi-inner product

\( \cos q \)

Q-I product and \( \cos q \)

Four point condition

Main result

Complete domains

2-roundness

2-roundness

Theorem

Gromov's problem

Curvature problem

Notation

Solution

Ptolemaic spaces

Bibliography

(UIUC, USA)
Examples

\[ S : z = \sqrt{|xy|}. \]

- \( S \) has zero curvature everywhere except for the origin, where it is \(-\infty\); \( S \) is a domain \( \mathbb{R}_0 \).
Let $\mathcal{M}$ be a non-empty set. Each ordered pair $(A, B) \in \mathcal{M} \times \mathcal{M}$ is called a *bound vector*. We will keep the notation $\overrightarrow{AB}$, or $\overrightarrow{u}$, for each bound vector $\overrightarrow{AB}$; the point $A$ is called the *tail* and the point $B$ is called the *head* of the bound vector $\overrightarrow{AB}$. 
The sum of two bound vectors \( \vec{u} \) and \( \vec{v} \) is defined if the head of \( \vec{u} \) coincides with the tail of \( \vec{v} \): if \( \vec{u} = \overrightarrow{AB} \) and \( \vec{v} = \overrightarrow{BC} \), then \( \vec{u} + \vec{v} = \overrightarrow{AC} \). An ordered pair of two bound vectors is called *admissible* if their sum is well defined.
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A *quasi-inner product* on a semimetric space \((\mathcal{M}, \rho)\) is a function \(\langle \ast, \ast \rangle : (\mathcal{M} \times \mathcal{M}) \times (\mathcal{M} \times \mathcal{M}) \to \mathbb{R}\) satisfying the following conditions:
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A *semimetric space* is a metric space without the triangle inequality axiom.

**Definition**

- A *quasi-inner product* on a semimetric space \((M, \rho)\) is a function \(\langle *, * \rangle : (M \times M) \times (M \times M) \to \mathbb{R}\) satisfying the following conditions:
  
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(iii) \(\langle -\vec{u}, \vec{v} \rangle = -\langle \vec{u}, \vec{v} \rangle\).
Quasi-inner product

- A **semimetric space** is a metric space without the triangle inequality axiom.

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  (iv) \(\langle \overrightarrow{u}, \overrightarrow{v} + \overrightarrow{w} \rangle = \langle \overrightarrow{u}, \overrightarrow{v} \rangle + \langle \overrightarrow{u}, \overrightarrow{w} \rangle\) if the pair \((\overrightarrow{v}, \overrightarrow{w})\) is admissible.
The quadrilateral cosine

\[ \cos q \left( \overrightarrow{PX}, \overrightarrow{QY} \right) = \frac{f^2 + g^2 - a^2 - b^2}{2xy}. \]
Lemma

Let \((\mathcal{M}, \rho)\) be a semimetric space with a quasi-inner product \(\langle *, * \rangle\) on it.
Lemma

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- Then, for every pair of bound vectors \(\overrightarrow{u} = PX\) and \(\overrightarrow{v} = QY\) in \(M\),
Lemma

Let \((\mathcal{M}, \rho)\) be a semimetric space with a quasi-inner product \(\langle *, * \rangle\) on it.

Then, for every pair of bound vectors \(\overrightarrow{u} = \overrightarrow{PX}\) and \(\overrightarrow{v} = \overrightarrow{QY}\) in \(\mathcal{M}\),

\[
\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \| \overrightarrow{u} \| \| \overrightarrow{v} \| \cos q (\overrightarrow{u}, \overrightarrow{v}),
\]

if \(\| \overrightarrow{u} \|\) and \(\| \overrightarrow{v} \|\) are positive and
Lemma

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  \langle \overrightarrow{u}, \overrightarrow{v} \rangle = \| \overrightarrow{u} \| \| \overrightarrow{v} \| \cos_q (\overrightarrow{u}, \overrightarrow{v}),
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  if \(\| \overrightarrow{u} \|\) and \(\| \overrightarrow{v} \|\) are positive and
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  \langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0, \text{ if either } \| \overrightarrow{u} \| \text{ or } \| \overrightarrow{v} \| \text{ vanishes.}
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Lemma

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Then, for every pair of bound vectors \(\overrightarrow{u} = \overrightarrow{PX}\) and \(\overrightarrow{v} = \overrightarrow{QY}\) in \(\mathcal{M}\),

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\langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0, \text{ if either } \| \overrightarrow{u} \| \text{ or } \| \overrightarrow{v} \| \text{ vanishes.}
\]

In addition, the cosq-product given by the foregoing formula is a quasi-inner product.
A *quasi-inner product space* is a semimetric space with a quasi-inner product on it.
The Cauchy-Schwarz inequality condition

- A quasi-inner product space is a semimetric space with a quasi-inner product on it.
- Quasilinearization for a semimetric space \((\mathcal{M}, \rho)\) is the quasi-inner product space of all bound vectors of \(\mathcal{M}\).
The Cauchy-Schwarz inequality condition

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- Quasilinearization for a semimetric space \((\mathcal{M}, \rho)\) is the quasi-inner product space of all bound vectors of \(\mathcal{M}\).
- Quasilinearization enables us to formulate a Cauchy-Schwarz inequality which forces the upper curvature bound of zero.
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*Quasilinearization* for a semimetric space \((\mathcal{M}, \rho)\) is the quasi-inner product space of all bound vectors of \(\mathcal{M}\).

Quasilinearization enables us to formulate a Cauchy-Schwarz inequality which forces the upper curvature bound of zero.

**The Cauchy-Schwarz inequality condition:**

\[ |\langle \overrightarrow{AB}, \overrightarrow{CD} \rangle| \leq \| \overrightarrow{AB} \| \| \overrightarrow{CD} \|, \]

for every quadruple \(\{A, B, C, D\} \subseteq \mathcal{M}\).
The four point condition

- The Cauchy-Schwarz inequality condition is equivalent to $\left| \cos q \left( \vec{AB}, \vec{CD} \right) \right| \leq 1$, for every pair of distinct points $(A, B)$ and $(C, D)$ in $\mathcal{M}$. 

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The four point condition:
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- The Cauchy-Schwarz inequality condition is equivalent to $|\cos q(\overrightarrow{AB}, \overrightarrow{CD})| \leq 1$, for every pair of distinct points $(A, B)$ and $(C, D)$ in $\mathcal{M}$.

- We readily see that $\cos q(\overrightarrow{AB}, \overrightarrow{CD}) = -\cos q(\overrightarrow{BA}, \overrightarrow{CD})$. Hence, the following condition is equivalent to the Cauchy-Schwarz inequality condition:
The four point condition

- The Cauchy-Schwarz inequality condition is equivalent to $|\cos q (\overrightarrow{AB}, \overrightarrow{CD})| \leq 1$, for every pair of distinct points $(A, B)$ and $(C, D)$ in $\mathcal{M}$.

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- The four point $\cos q$ condition:
  $\cos q (\overrightarrow{AB}, \overrightarrow{CD}) \leq 1$, for every pair of distinct points $(A, B)$ and $(C, D)$ in $\mathcal{M}$. 
Main result

- Recall that a metric space is *geodesically connected* if any pair of its points can be joined by a shortest.

Theorem
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**Theorem**

- *A geodesically connected metric space is an $\mathcal{R}_0$ domain if and only if it satisfies the four point \cosq condition.*
Main result

- Recall that a metric space is *geodesically connected* if any pair of its points can be joined by a shortest.

**Theorem**

- A *geodesically connected metric space is an* $\mathcal{R}_0$ *domain if and only if it satisfies the four point* $\cos q$ *condition.

- If for a quadruple $\{A, B, C, D\}$, $A \neq B$, $C \neq D$, in a geodesically connected metric space with the four point $\cos q$ condition, $\cos q\left(\overrightarrow{AB}, \overrightarrow{CD}\right) = 1$, then the geodesic convex hull of the set $\{A, B, C, D\}$ is either isometric to a trapezoid in the Euclidean plane or a segment of straight line.
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**Definition**
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It is well known that a convergent sequence in a semimetric space need not be a Cauchy sequence.

**Definition**
Cauchy condition and weak convexity in a semimetric space

- Cauchy sequences in a semimetric space are defined in the same way as in a metric space.
- It is well known that a convergent sequence in a semimetric space need not be a Cauchy sequence.

**Definition**

- A semimetric space is said to be *weakly convex* if, for every $A, B \in \mathcal{M}$ there is $\lambda$, $0 < \lambda < 1$, such that, for every $\varepsilon > 0$, there is $C_\varepsilon \in \mathcal{M}$ satisfying the inequalities
  \[ |\rho (A, C_\varepsilon) - \lambda \rho (A, B)| < \varepsilon \quad \text{and} \quad |\rho (B, C_\varepsilon) - (1 - \lambda) \rho (A, B)| < \varepsilon. \]
Characterization of complete Aleksandrov domains

Theorem

- A semimetric space $(M, \rho)$ is isometric to a complete $\mathcal{R}_0$ domain if and only if the following conditions are satisfied:
Characterization of complete Aleksandrov domains

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- A semimetric space $(\mathcal{M}, \rho)$ is isometric to a complete $\mathcal{R}_0$ domain if and only if the following conditions are satisfied:

(i) $(\mathcal{M}, \rho)$ is weakly convex.
Characterization of complete Aleksandrov domains

Theorem

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(i) \((M, \rho)\) is weakly convex.

(ii) Each Cauchy sequence in \((M, \rho)\) has a limit.
Characterization of complete Aleksandrov domains

Theorem

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(i) \((M, \rho)\) is weakly convex.

(ii) Each Cauchy sequence in \((M, \rho)\) has a limit.

(iii) \((M, \rho)\) satisfies the four point \(\cos q\) condition.
Enflows roundness condition

Definition

- A metric space \((\mathcal{M}, \rho)\) is said to have *roundedness* \(p\) if \(p\) is the supremum of the set of \(q\) with the property:
Enflows roundness condition

Definition

- A metric space \((\mathcal{M}, \rho)\) is said to have roundedness \(p\) if \(p\) is the supremum of the set of \(q\) with the property:

- for every quadruple of points \(Q = \{P_1, P_2, P_3, P_4\} \subset \mathcal{M}\),

\[
\rho_q(P_1, P_3) + \rho_q(P_2, P_4) + \rho_q(P_3, P_4) + \rho_q(P_4, P_1) \geq \rho_q(P_1, P_2) + \rho_q(P_2, P_3).
\]
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  - for every quadruple of points \(Q = \{P_1, P_2, P_3, P_4\} \subset \mathcal{M}\),

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\rho^q (P_1, P_3) + \rho^q (P_2, P_4) 
\leq \rho^q (P_1, P_2) + \rho^q (P_2, P_3) 
+ \rho^q (P_3, P_4) + \rho^q (P_4, P_1).
\]
P. Enflo applied the notion of $p$-roundness to the study of uniformly continuous homeomorphisms between Banach spaces.
Remarks

- P. Enflo applied the notion of $p$-roundness to the study of uniformly continuous homeomorphisms between Banach spaces.
- By the triangle inequality the inequality in the definition of $p$-roundness holds for every quadruple if $q = 1$. 
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It is known, that for a geodesically connected metric space, the same inequality does not hold for every quadruple if $q > 2$. 
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By the triangle inequality the inequality in the definition of \( p \)-roundness holds for every quadruple if \( q = 1 \).

It is known, that for a geodesically connected metric space, the same inequality does not hold for every quadruple if \( q > 2 \).

Hence, for a geodesically connected metric space, \( p \) must satisfy the inequality \( 1 \leq p \leq 2 \).
Enflo's two-roundness condition

For every quadruple \( Q = \{P_1, P_2, P_3, P_4\} \),

\[
\rho^2(P_1, P_3) + \rho^2(P_2, P_4) + \rho^2(P_1, P_2) + \rho^2(P_2, P_3).
\]

In \( \mathbb{R}^2 \), this inequality is due to L. Euler (1750).
For every quadruple \( Q = \{ P_1, P_2, P_3, P_4 \} \),
\[
\rho^2 (P_1, P_3) + \rho^2 (P_2, P_4) \leq \rho^2 (P_1, P_2) + \rho^2 (P_2, P_3) + \rho^2 (P_3, P_4) + \rho^2 (P_4, P_1).
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For every quadruple \( Q = \{P_1, P_2, P_3, P_4\} \),
\[
\rho^2(P_1, P_3) + \rho^2(P_2, P_4) \leq \rho^2(P_1, P_2) + \rho^2(P_2, P_3) + \rho^2(P_3, P_4) + \rho^2(P_4, P_1).
\]
In \( \mathbb{R}^2 \), this inequality is due to L. Euler (1750).
Enflo’s roundness condition and Aleksandrov domains

Theorem

- A geodesically connected metric space is an $\mathcal{R}_0$ domain if and only if it satisfies Enflo’s 2-roundness condition.
Enflo’s roundness condition and Aleksandrov domains

Theorem

- A geodesically connected metric space is an $R_0$ domain if and only if it satisfies Enflo’s 2-roundness condition.

- In addition, if $(\mathcal{M}, \rho)$ is a geodesically connected metric space with Enflo’s 2-roundness condition and $Q = \{A, B, C, D\}$ is a quadruple of distinct points in $\mathcal{M}$ for which $\rho^2(A, C) + \rho^2(B, D) = \rho^2(A, B) + \rho^2(B, C) + \rho^2(C, D) + \rho^2(A, D)$, then the geodesic convex hull of the set $\{A, B, C, D\}$ is either isometric to a parallelogram in the Euclidean plane or a segment of straight line.
Let $r \in \mathbb{N}$ and $M_r$ denote the set of all symmetric $r \times r$ matrices with zero diagonal entries and non-negative entries otherwise.
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Let \((\mathcal{X}, d)\) be a semimetric space. Then \( K_r(\mathcal{X}) \) consists of all matrices \( A = (a_{ij}) \) in \( M_r \) such that for every \( A \in K_r(\mathcal{X}) \) there is an \( r \)-tuple \( \{P_1, P_2, ..., P_r\} \subseteq \mathcal{X} \) satisfying \( a_{ij} = d(P_i, P_j) \), \( i, j = 1, 2, ..., r \).
K-curvature classes

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- A subset \( \mathcal{K} \subseteq M_r \) defines Gromov's (global) \( \mathcal{K} \)-curvature class as follows:
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We will need the following notations:
Define the following two classes $\mathcal{K}$ of subspaces of $M_4$:

$$
\mathcal{K}_{\cosq} = \{ (a_{ij})_{i,j=1,2,3,4} \in M_4 | a_{21}3 + a_{22}4 + a_{23}2 + a_{24}1 + a_{34}2 \}.
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Notation

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$$\mathcal{K}_E = \left\{ (a_{ij})_{i,j=1,2,3,4} \in M_4 \mid a_{13}^2 + a_{24}^2 \leq a_{23}^2 + a_{14}^2 + a_{12}^2 + a_{34}^2 \right\}.$$
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Notation

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- Let $\mathcal{M}_S$ denote the set of all semimetric spaces satisfying conditions:
  
  (i) $(\mathcal{M}, \rho)$ is weakly convex.
  
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- The following theorem gives a complete solution of the Gromov curvature problem in the context of Aleksandrov spaces of non-positive curvature.
Solution of Gromov’s curvature problem

Theorem

\((\mathcal{X}, \rho) \in \mathcal{M}_G\) is in the \(\mathcal{K}_{\cos q}\)-curvature class if and only if \((\mathcal{X}, \rho)\) is isometric to an \(\mathcal{R}_0\) domain.
Solution of Gromov’s curvature problem

Theorem

- $(X, \rho) \in \mathcal{M}_G$ is in the $\mathcal{K}_{\cos q}$-curvature class if and only if $(X, \rho)$ is isometric to an $\mathcal{R}_0$ domain.
- $(X, \rho) \in \mathcal{M}_G$ is in the $\mathcal{K}_E$-curvature class if and only if $(X, \rho)$ is isometric to an $\mathcal{R}_0$ domain.
Solution of Gromov’s curvature problem

Theorem

1. \((X, \rho) \in \mathcal{M}_G\) is in the \(\mathcal{K}_{\cos q}\)-curvature class if and only if \((X, \rho)\) is isometric to an \(\mathbb{R}_0\) domain.

2. \((X, \rho) \in \mathcal{M}_G\) is in the \(\mathcal{K}_E\)-curvature class if and only if \((X, \rho)\) is isometric to an \(\mathbb{R}_0\) domain.

3. \((X, \rho) \in \mathcal{M}_G\) is in the \(\mathcal{K}_{\cos q}\)-curvature class if and only if \((X, \rho)\) is isometric to a complete \(\mathbb{R}_0\) domain.
A semimetric space \((\mathcal{M}, \rho)\) is called *Ptolemaic* if for each quadruple \(\{P_1, P_2, P_3, P_4\} \subset \mathcal{M}\),

\[
\rho(P_1, P_4) \rho(P_2, P_3) \leq \rho(P_1, P_2) \rho(P_3, P_4) + \rho(P_1, P_3) \rho(P_2, P_4).
\]
Ptolemaic spaces

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\]

- In 1952, I.J. Shoenberg has shown that a semi-normed space is Ptolemaic if and only if it is an inner product space.
Ptolemaic spaces

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- In 1963, D.C. Kay showed that a Riemannian manifold is non-positively curved if and only if it is locally Ptolemaic.
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We prove: An $\mathcal{R}_0$ domain is Ptolemaic.
Bibliography


