Parametrizations and Integrals

1. Consider the ellipsoid with implicit equation
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \]
   (a) Parameterize this ellipsoid.
   (b) Set up, but do not evaluate, a double integral that computes its surface area.

2. Let \( \mathbf{r}(u, v) = \langle (2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u \rangle \),
   where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \).
   (a) Sketch the surface parameterized by this function.
   (b) Compute its surface area.

3. Consider the surface integral \( \iint_{\Sigma} z \, dS \)
   where \( \Sigma \) is the surface with sides \( S_1 \) given by the cylinder \( x^2 + y^2 = 1 \), \( S_2 \) given by the unit disk in the \( xy \)-plane, and \( S_3 \) given by the plane \( z = x + 1 \). Evaluate this integral as follows:
   (a) Parameterize \( S_1 \) using \( (\theta, z) \) coordinates.
   (b) Evaluate the integral over the surface \( S_2 \) without parameterizing.
   (c) Parameterize \( S_3 \) in (Des)cartesian coordinates and evaluate the resulting integral using polar coordinates.

4. Let \( C \) be the circle in the plane with equation \( x^2 + y^2 - 2x = 0 \).
   (a) Parameterize \( C \) as follows. For each choice of a slope \( t \), consider the line \( L_t \) whose equation is \( y = tx \). Then the intersection \( L_t \cap C \) of \( L_t \) and \( C \) contains two points, one of which is \( (0, 0) \). Find the other point of intersection, and call its \( x \)- and \( y \)-coordinates \( x(t) \) and \( y(t) \). Compute a formula for \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \). Check your answer with your TA.
   (b) Suppose that \( t = \frac{p}{q} \) is a rational number. Show that \( x(p/q) \) and \( y(p/q) \) are also rational numbers. Explain how, by clearing denominators in \( x(p/q) - 1 \) and \( y(p/q) \), you can find a triple of integers \( U, V, \) and \( W \) for which \( U^2 + V^2 = W^2 \).
   (c) Compute \( \int_{C} \frac{1}{2} \langle -y, x \rangle \cdot d\mathbf{r} \) using your parameterization above.