1. Let $S$ be the portion of the cylinder of radius 2 about the $x$-axis where $-1 \leq x \leq 1$.

   (a) Draw a picture of $S$ and compute its area without doing any integrals. Hint: How could you make this cylinder out of paper?
   **SOLUTION:** We have $\text{Area}(S) = 8\pi$ since we can cut $S$ along a line in the $xy$-plane and flatten it out into a $2 \times 4\pi$ rectangle.

   (b) Find a parameterization $\mathbf{r}(u, v)$ of $S$.
   **SOLUTION:**
   Let $u = x$ and $v = \text{angle about the } x - \text{axis}$, so that
   $$\mathbf{r}(u, v) = (u, 2\cos v, 2\sin v)$$
   with domain $D = \{-1 \leq u \leq 1 \text{ and } 0 \leq v \leq 2\pi\}$.

   (c) Does the normal vector field associated to your parameterization point into or out of $S$? First, try to determine this without doing any calculations, and then check your answer by evaluating $\mathbf{r}_u \times \mathbf{r}_v$.
   **SOLUTION:**
   By the right hand rule, the vector points inwards. Check:
   $$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -2\sin v & 2\cos v \end{vmatrix} = \langle 0, -2\cos v, -2\sin v \rangle$$
   At $u = 0, v = 0$ we have $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -2, 0 \rangle$ which points inwards.

   (d) If necessary, change your parameterization so that the normal vector field points *inwards*.
   **SOLUTION:**
   If your vector points outwards, interchanging the role of $u$ and $v$ will reverse its direction.

   (e) Now consider the vector field $\mathbf{F} = \langle -z, xz, -xy \rangle$. Compute $\text{curl}\mathbf{F}$.
   **SOLUTION:**
   $\text{curl}\mathbf{F} = \langle -2x, y - 1, z \rangle$
(f) Check that \( \text{curl} \mathbf{F} \) is the sum of \( \mathbf{G} = \langle -2x, -1, 0 \rangle \) and \( \mathbf{H} = \langle 0, y, z \rangle \).

**SOLUTION:**
\[
\mathbf{G} + \mathbf{H} = \langle -2x, -1, 0 \rangle + \langle 0, y, z \rangle = \langle -2x, y - 1, z \rangle = \text{curl} \mathbf{F}
\]

(g) Use geometric arguments to determine whether the flux of \( \mathbf{G} \) is positive, zero, or negative. Remember that we have oriented \( S \) so that the normals point inwards. Do the same for \( \mathbf{H} \) and \( \text{curl} \mathbf{F} \).

**SOLUTION:**
Since every normal vector to \( S \) has 0 \( x \)-component, the flux of \( \mathbf{G} \) is the same as the flux of \( \langle 0, -1, 0 \rangle \), which by symmetry must be equal to 0. For \( \mathbf{H} \), at each point of \( S \) \( \mathbf{H} \) points outward from the surface, so the flux is negative (since the normals point inwards). In fact, \( \mathbf{H} = -2\mathbf{n} \).

(h) Using your parametrization, directly compute the flux of \( \text{curl} \mathbf{F} \).

**SOLUTION:**
\[
\int \int_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dA = \int_{-1}^{1} \int_{0}^{2\pi} \langle -2u, 2\cos v - 1, 2\sin v \rangle \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dv \, du = \int_{-1}^{1} \int_{0}^{2\pi} -4(\cos^2 v + \cos v \sin^2 v) \, dv \, du = 4 \int_{-1}^{1} -1 + \cos v \, dv \, du = 4 \int_{-1}^{1} -2 \pi \, du = -16\pi
\]

(i) Check your answer in (h) using Stokes’ Theorem. Note here that \( \partial S \) has two boundary components, and make sure that you orient them correctly.

**SOLUTION:**
The boundary components should be oriented like so:

Parametrize \( C_1 \) by \( \langle 1, 2\cos t, 2\sin t \rangle \) for \( 0 \leq t \leq 2\pi \) and \( C_2 \) by \( \langle -1, 2\cos t, -2\sin t \rangle \) for \( 0 \leq t \leq 2\pi \). Now
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} \langle -2\sin t, 2\sin t, -2\cos t \rangle \cdot \langle 0, -2\sin t, 2\cos t \rangle \, dt = \int_{0}^{2\pi} -4(\sin t^2 + \cos^2 t) \, dt = -8\pi.
\]
Similarly, \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -8\pi \). Thus \( \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = -8\pi + (-8\pi) = -16\pi = \int \int_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dA \) as guaranteed by Stokes’ theorem.
(j) Check your answer in (h) a second time by using what you learned in (g) to compute the flux of \( \mathbf{G} \) and \( \mathbf{H} \).

**SOLUTION:**
We know \( \text{curl} \mathbf{F} = \mathbf{G} + \mathbf{H} \) and the flux of \( \mathbf{G} \) through \( S \) is 0. Since \( \mathbf{H} = -2\mathbf{n}, \mathbf{H} \cdot \mathbf{n} = -2 \). Hence

\[
\int \int_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} dA = \int \int_S \mathbf{H} \cdot \mathbf{n} dA = \int \int_S -2 dA = -2 \text{Area}(S) = -2 \cdot 8\pi = -16\pi
\]

2. Consider the surface \( S \) shown below, which is oriented using the outward pointing normal.

(a) Suppose \( \mathbf{F} \) is a vector field on \( \mathbb{R}^3 \) which is equal to \( \text{curl} \mathbf{G} \) for some unknown vector field \( \mathbf{G} \). Suppose the line integral of \( \mathbf{G} \) around the unit circle (oriented counter-clockwise) in the \( xy \)-plane is 25. Determine the flux of \( \mathbf{F} \) through \( S \).

**SOLUTION:**
By Stokes: \( \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{G} \cdot d\mathbf{r} = 25 \).

(b) Suppose \( \mathbf{H} \) is a vector field on \( \mathbb{R}^3 \) which is equal to \( \text{curl} \mathbf{B} \) for some unknown vector field \( \mathbf{B} \). If \( \mathbf{H}(x, y, 0) = \mathbf{k} \), find the flux of \( \mathbf{H} \) through the surface \( S \).

**SOLUTION:**
Let \( D \) be the unit disc with upwards normal \( \mathbf{n} = \mathbf{k} = \langle 0, 0, 1 \rangle \) Then \( S \) and \( D \) have the same oriented boundary, the counter-clockwise oriented unit circle \( C \). Hence by Stokes:

\[
\int \int_S \mathbf{H} \cdot d\mathbf{r} = \int_C \mathbf{B} \cdot d\mathbf{r} = \int \int_D \mathbf{H} \cdot d\mathbf{R} = \int \int_D \mathbf{k} \cdot k dA = \text{Area}(D) = \pi.
\]