Green’s Theorem

Green’s Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus: it relates the (integral of) a vector field \( \mathbf{F} \) on the boundary of a region \( D \) to the integral of a suitable derivative of \( \mathbf{F} \) over the whole of \( D \).

1. Let \( D \) be the unit square with vertices (0,0), (1,0), (0,1), and (1,1) and consider the vector field \( \mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x + y \rangle \). See below right for a plot.

   (a) For the curve \( C = \partial D \) oriented counterclockwise, directly evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \). Hint: to speed things up, have each group member focus on one side of \( C \).

   (b) Now compute \( \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \).

   (c) Check that Green’s Theorem works in this example.

**SOLUTION:**

(a) We split the square up into four pieces, parametrizing and integrating one at a time. 

Right side: \( C_1 \) is parametrized by \( \mathbf{r}_1(t) = (1, t), 0 \leq t \leq 1 \).

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t, 1 + t) \cdot (0, 1) \, dt = \int_0^1 1 + td\, t = \left[ t + \frac{1}{2} t^2 \right]_0^1 = \frac{3}{2}
\]

Top side: \( C_2 \) is parametrized by \( \mathbf{r}_2(t) = (1 - t, 1), 0 \leq t \leq 1 \).

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1 - t, -t) \cdot (-1, 0) \, dt = \int_0^1 t - 1 \, dt = \left[ \frac{1}{2} t^2 - t \right]_0^1 = -\frac{1}{2}
\]

Left side: \( C_3 \) is parametrized by \( \mathbf{r}_3(t) = (0, 1 - t), 0 \leq t \leq 1 \).

\[
\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, 1 - t) \cdot (0, -1) \, dt = \int_0^1 -t - 1 \, dt = \left[ \frac{1}{2} t^2 - t \right]_0^1 = -\frac{1}{2}
\]

Bottom side: \( C_4 \) is parametrized by \( \mathbf{r}_4(t) = (t, 0), 0 \leq t \leq 1 \).

\[
\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, t) \cdot (1, 0) \, dt = \int_0^1 0 \, dt = 0
\]

So, the line integral around the entire boundary \( C \) going counterclockwise is

\[
\int_{C = \partial D} \mathbf{F} \cdot d\mathbf{r} = \frac{3}{2} - \frac{1}{2} - \frac{1}{2} + 0 = \frac{1}{2}
\]

(b) \[
\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_0^1 \int_0^1 1 - x \, dx \, dy = \int_0^1 \left[ x - \frac{1}{2} x^2 \right]_{x=0}^{x=1} dy = \int_0^1 \frac{1}{2} \, dy = \frac{1}{2}
\]
(c) The result for either computation was $\frac{1}{2}$, demonstrating Green’s theorem for this example.

2. Compute the line integral of $F(x,y) = \langle x^3, 4x \rangle$ along the path $C$ shown at right against a grid of unit-sized squares. To save work, use Green’s Theorem to relate this to a line integral over the vertical path joining $B$ to $A$. Hint: Look at the region $D$ bounded by these two paths. Check your answer with the instructor.

**SOLUTION:**

Let $L$ be the line segment going from $B$ to $A$. Then, we can now apply Green’s theorem to combination of $C$ and $L$. Let $D$ be the region bounded by these two paths. Then, by Green’s theorem, since we are oriented correctly,

$$\int_{\partial D} F \cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA = \iint_D \frac{\partial}{\partial x} (4x) - \frac{\partial}{\partial y} (x^3) \ dA = \iint_D 4 \ dA = 4 \cdot \text{Area}(A) = 16$$

because the area of the region is made of exactly 4 unit squares. The boundary of $D$ is $C$ and $L$:

$$\int_{\partial D} F \cdot dr = \int_C F \cdot dr + \int_L F \cdot dr = 16$$

The line integral along $L$ is easier: parametrizing $L$ by $r(t) = (-1, t)$ for $-1 \leq t \leq 0$, we get

$$\int_L F \cdot dr = \int_{-1}^{0} (-1, -4) \cdot (0, 1) \ dt = \int_{-1}^{0} -4 \ dt = -4$$

Putting it together,

$$\int_C F \cdot dr = 16 - \int_L F \cdot dr = 16 - (-4) = 20$$
3. Consider the quarter circle $C$ shown below and the vector field $\mathbf{F}(x, y) = \langle 2xe^y, x + x^2e^y \rangle$. The goal of this problem is to compute the line integral $I_0 = \int_C \mathbf{F} \cdot d\mathbf{r}$.

(a) Parameterize $C$ and start directly expanding out $I_0$ into an ordinary integral in $t$ until you are convinced that finding $I_0$ this way will be a highly unpleasant experience.

(b) Check that $\mathbf{F}$ is not conservative, so we can’t use that trick directly to compute $I_0$.

(c) Find a function $f(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla f$, where $\mathbf{G}$ is the vector field $\langle 0, x \rangle$.

(d) Argue geometrically that $\mathbf{G}$ integrates to 0 along any line segment contained in either the $x$-axis or the $y$-axis.

(e) Use part (d) with Green’s Theorem to show that $\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$.

(f) Combine parts (c–e) with the Fundamental Theorem of Line Integrals to evaluate $I_0$. Check your answer with the instructor.

**SOLUTION:**

(a) 

(b) Look at the partials: $\frac{\partial P}{\partial y} = 2xe^y \neq 1 + 2xe^y = \frac{\partial Q}{\partial x}$. For conservative vector fields, these two partials have to be the same (since mixed partial commute).

(c) $\mathbf{F} - \mathbf{G} = \langle 2xe^y, x^2e^y \rangle$, which now is conservative, with potential function $f(x, y) = x^2e^y$

(d) Along the $y$-axis, $x = 0$, so $\mathbf{G} = \langle 0, 0 \rangle$. Along the $x$-axis, $\mathbf{G} = \langle 0, x \rangle$, is always exactly vertical, hence perpendicular to any portion of the $x$-axis. So, the line integral will be 0.

(e) Take $D$ to be region bounded by the arc $C$ and the two axes. By part (d),

$$\int_{\partial D} \mathbf{G} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r}$$

Now, Green’s Theorem tells us that

$$\int_{\partial D} \mathbf{G} \cdot d\mathbf{r} = \iint_D 1 \, dA = \frac{1}{4}(\pi \cdot 4^2) = 4\pi$$

Combining these tells us that

$$\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$$
(f) \( \mathbf{F} = \mathbf{G} + \nabla f \), so taking the line integrals,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r} + \int_C \nabla f \cdot d\mathbf{r}
\]

In (e), we computed the first integral, and we can immediately evaluate the other using the Fundamental Theorem of Calculus for Line Integrals:

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi + (f(0, 4) - f(4, 0)) = 4\pi - 16
\]

4. Consider the shaded region \( V \) shown, bounded by a circle \( C_1 \) of radius 5 and two smaller circles \( C_2 \) and \( C_3 \) of radius 1. Suppose \( \mathbf{F}(x, y) = \langle P, Q \rangle \) is a vector field where \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 \) on \( V \). Assuming in addition that \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi \) and \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi \), compute \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \). Check your answer with the instructor.

**SOLUTION:**

Note that the outer curve \( C_1 \) is oriented as we would want it to be to use for Green’s theorem, but the other two are oriented backwards (walking along \( C_2 \) or \( C_3 \), our left arm points outside \( V \)). So, \( \partial V = C_1 - C_2 - C_3 \), and Green’s theorem gives

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \iint_V \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = 3\pi + 4\pi + 2 \cdot \text{Area}(V)
\]

The area is just that of the larger disk minus the other two:

\[
\text{Area}(V) = \pi \cdot 5^2 - \pi - \pi = 23\pi
\]

and so

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 53\pi
\]

5. Suppose \( D \) is a region in the plane bounded by a closed curve \( C \). Use Green’s Theorem to show that both \( \int_C x \, dy \) and \( -\int_C y \, dx \) are equal to \( \text{Area}(D) \).

**SOLUTION:**

Using the alternate notation for line integrals, Green’s theorem says

\[
\int_{\partial D} P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA
\]

So, applying this to the given vector fields:

\[
\int_{\partial D} x \, dy = \iint_D \frac{\partial}{\partial x} x \, dA = \iint_D 1 \, dA = \text{Area}(D)
\]

\[
-\int_{\partial D} y \, dx = -\iint_D \frac{\partial}{\partial y} y \, dA = \iint_D 1 \, dA = \text{Area}(D)
\]
6. The curve satisfying \( x^3 + y^3 = 3xy \) is called the Folium of Descartes and is shown at right.

(a) Let \( C \) be the “bulb” part of this folium, more precisely, the part in the positive quadrant. Show that any line \( y = tx \) for \( t > 0 \) meets \( C \) in exactly two points, one of which is the origin. Use this fact to parameterize \( C \) by taking the slope \( t \) as the parameter.

(b) Use part (a) and Problem 5 to compute the area bounded by \( C \). Check your answer with the instructor.

**SOLUTION:**

(a) Our curve is defined by \( x^3 + y^3 - 3xy = 0 \). If we consider a line of slope \( t > 0 \) through the origin, it meets \( C \) in one other point. Let \( y = tx \); then,

\[
x^3 + t^3x^3 - 3tx^2 = 0
\]

\[
x^2(x + t^3x - 3t) = 0
\]

We can divide by \( x \) as we are looking for the solution with \( x > 0 \):

\[
x = \frac{3t}{1 + t^3} \implies y = \frac{3t^2}{1 + t^3}
\]

This gives a parametrization for \( 0 \leq t < \infty \).

(b) We use the first integral in the previous problem to compute the area:

\[
\text{Area}(D) = \int_C x \, dy = \int_0^\infty \frac{3t}{1 + t^3} \left( \frac{6t(1 + t^3) - 3t^2(3t^2)}{(1 + t^3)^2} \right) \, dt = \int_0^\infty \frac{3t^2(6 - 3t^2)}{(1 + t^3)^3} \, dt
\]

We can compute this integral with a \( u \)-substitution, with \( u = 1 + t^3 \), \( du = 3t^2 \):

\[
\int_1^\infty \frac{9 - 3u}{u^3} \, du = \int_1^\infty 9u^{-3} - 3u^{-2} \, du = \left[ -\frac{9}{2}u^{-2} + 3u^{-1} \right]_{u=1}^{u=\infty} = 0 - \left( -\frac{9}{2} + 3 \right) = \frac{3}{2}
\]