1. Consider the ellipsoid with implicit equation
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \]

(a) Parametrize this ellipsoid.

**Solution.** One could use the parametrization
\[ x = a \sin \phi \cos \theta, \quad y = b \sin \phi \sin \theta, \quad z = c \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi. \]

(b) Set up, but do not evaluate, a double integral that computes its surface area.

**Solution.** Since \( \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta, b \sin \phi \sin \theta, c \cos \phi) \), one has
\[ \mathbf{r}_\phi = (a \cos \phi \cos \theta, b \cos \phi \sin \theta, -c \sin \phi), \quad \mathbf{r}_\theta = (-a \sin \phi \sin \theta, b \sin \phi \cos \theta, 0), \]
so
\[ \mathbf{r}_\phi \times \mathbf{r}_\theta = (bc \sin^2 \phi \cos \theta, ac \sin^2 \phi \sin \theta, ab \sin \phi \cos \theta). \]
Therefore
\[ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi}, \]
and the surface area is computed by
\[ \text{Area} = \int_0^{2\pi} \int_0^\pi |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, d\phi d\theta \]
\[ = \int_0^{2\pi} \int_0^\pi \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi} \, d\phi d\theta. \]

2. Let
\[ \mathbf{r}(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u), \]
where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \).

(a) Sketch the surface parametrized by this function.

**Solution.** The sketch of the surface is as follows.
Compute its surface area.

**Solution.** By the parametrization, one has
\[ r_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \]
\[ r_v = (- (2 + \cos u) \sin v, (2 + \cos u) \cos v, 0), \]
and so
\[ r_u \times r_v = (- (2 + \cos u) \cos u \cos v, -(2 + \cos u) \cos u \sin v, -(2 + \cos u) \sin u). \]

Therefore \(|r_u \times r_v| = 2 + \cos u\), and the surface area is computed by
\[ \text{Area} = \int_0^{2\pi} \int_0^{2\pi} |r_u \times r_v| \, du \, dv = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos u) \, du \, dv = 8\pi^2. \]

3. Consider the surface integral
\[ \iiint_{\Sigma} z \, dS \]
where \(\Sigma\) is the surface with sides \(S_1\) given by the cylinder \(x^2 + y^2 = 1\), \(S_2\) given by the unit disk in the \(xy\)-plane, and \(S_3\) given by the plane \(z = x + 1\). Evaluate this integral as follows:

(a) Parametrize \(S_1\) using \((\theta, z)\) coordinates.

**Solution.** One can parametrize \(S_1\) by
\[ x = \cos \theta, \ y = \sin \theta, \ z = z, \quad 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq \cos \theta + 1. \]

(b) Evaluate the integral over the surface \(S_2\) without parametrizing.

**Solution.** Since \(z = 0\) on \(S_2\), we know \(\iint_{S_2} z \, dS = 0\).
(c) Parametrize \(S_3\) in Cartesian coordinates and evaluate the resulting integral using polar coordinates.

**Solution.** One can parametrize \(S_3\) in Cartesian coordinates
\[
x = x, \quad y = y, \quad z = x + 1, \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.
\]

Now we move to evaluate the integral \(\iiint_S z \, dS\). Obviously
\[
\iiint_S z \, dS = \iiint_{S_1} z \, dS + \iiint_{S_2} z \, dS + \iiint_{S_3} z \, dS := I_1 + I_2 + I_3.
\]

To estimate \(I_1\), using the parametrization in (a), one has
\[
r(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle.
\]
Then
\[
r_\theta = \langle -\sin \theta, \cos \theta, 0 \rangle, \quad r_z = \langle 0, 0, 1 \rangle,
\]
and
\[
r_\theta \times r_z = \langle \cos \theta, \sin \theta, 0 \rangle.
\]
So \(|r_\theta \times r_z| = 1\), and
\[
I_1 = \int_0^{2\pi} \int_0^{\cos \theta + 1} z \, dz \, d\theta = \int_0^{2\pi} \frac{(\cos \theta + 1)^2}{2} \, d\theta
\]
\[
= \int_0^{2\pi} \frac{\cos^2 \theta + 2 \cos \theta + 1}{2} \, d\theta = \frac{3\pi}{2}.
\]

In (b) we know \(I_2 = 0\). To evaluate \(I_3\), by the parametrization in (c), one has
\[
r(x, y) = \langle x, y, x + 1 \rangle, \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},
\]
and so
\[
r_x = \langle 1, 0, 1 \rangle, \quad r_y = \langle 0, 1, 0 \rangle, \quad r_x \times r_y = \langle -1, 0, 1 \rangle.
\]
Thus \(|r_x \times r_y| = \sqrt{2}\), and the surface integral is
\[
I_3 = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + 1) \sqrt{2} \, dy \, dx = \int_{\sqrt{x^2+y^2}\leq 1} (x + 1) \sqrt{2} \, dy \, dx.
\]
To evaluate this integral, one can use the polar coordinates
\[
x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.
\]
Therefore,
\[
I_3 = \int_0^{2\pi} \int_0^1 (r \cos \theta + 1) \sqrt{2} \, r \, dr \, d\theta = \sqrt{2}\pi.
\]
Adding up all three integrals, one gets
\[
\iiint_S z \, dS = I_1 + I_2 + I_3 = \frac{3\pi}{2} + \sqrt{2}\pi.
\]
4. Let $C$ be the circle in the plane with equation $x^2 + y^2 - 2x = 0$.

(a) Parametrize $C$ as follows. For each choice of a slope $t$, consider the line $L_t$ whose equation is $y = tx$. Then the intersection $L_t \cap C$ of $L_t$ and $C$ contains two points, one of which is $(0, 0)$. Find the other point of intersection, and call its $x-$ and $y-$coordinates $x(t)$ and $y(t)$. Compute a formula for $r(t) = \langle x(t), y(t) \rangle$. Check your answer with your TA.

**Solution.** Bring $y = tx$ into $x^2 + y^2 - 2x = 0$, then one has $x^2 + t^2x^2 - 2x = 0$, and it is easy to get $x = \frac{2}{1+t^2}$, and then $y = \frac{2t}{1+t^2}$. Thus $r(t) = \langle \frac{2}{1+t^2}, \frac{2t}{1+t^2} \rangle$.

(b) Suppose that $t = \frac{p}{q}$ is a rational number. Show that $x(p/q)$ and $y(p/q)$ are also rational numbers. Explain how, by clearing denominators in $x(p/q) - 1$ and $y(p/q)$, you can find a triple of integers $U, V, W$ for which $U^2 + V^2 = W^2$.

**Solution.** Plug $t = \frac{p}{q}$ into the the parametrization, one gets

$$x(p/q) = \frac{2q^2}{p^2 + q^2}, \quad y(p/q) = \frac{2pq}{p^2 + q^2},$$

and both of them are rational numbers. Since $(x-1)^2 + y^2 = 1$, and $x(p/q) - 1 = \frac{q^2 - p^2}{p^2 + q^2}$, then one has

$$\left( \frac{q^2 - p^2}{p^2 + q^2} \right)^2 + \left( \frac{2pq}{p^2 + q^2} \right)^2 = 1.$$

By setting

$$U = q^2 - p^2, \quad V = 2pq, \quad W = p^2 + q^2,$$

one has $U^2 + V^2 = W^2$.

(c) Compute $\int_C \frac{1}{2} \langle -y, x \rangle \cdot dr$ using your parametrization above.

**Solution.** Since $r = \langle -\frac{2}{1+t^2}, \frac{2t}{1+t^2} \rangle$, one has $r' = \langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \rangle$. Then

$$\int_C \frac{1}{2} \langle -y, x \rangle \cdot dr = \int_\infty^{-\infty} \frac{1}{2} \langle -\frac{2t}{1+t^2}, \frac{2}{1+t^2} \rangle \cdot \langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \rangle dt$$

$$= \int_\infty^{-\infty} \frac{2}{(1+t^2)^2} \ dt = \pi.$$