Thursday, October 29   * Solutions   * Transformations of $\mathbb{R}^2$.

**Purpose:** In class, we’ve seen several different coordinate systems on $\mathbb{R}^2$ and $\mathbb{R}^3$ beyond the usual rectangular ones: polar, cylindrical, and spherical. The lectures on Friday and Monday will cover the crucial technique of simplifying hard integrals using a change of coordinates (Section 15.9). The point of this worksheet is to familiarize you with some basic concepts and examples for this process.

**Starting point:** Here we consider a variety of transformations $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Previously, we have used such functions to describe vector fields on the plane, but we can also use them to describe ways of distorting the plane:

1. Consider the transformation $T(x, y) = (x - 2y, x + 2y)$.

   (a) Compute the image under $T$ of each vertex in the below grid and make a careful plot of them, which should be fairly large as you will add to it later.

   To speed this up, divide the task up among all members of the group.

   ![Image of grid and transformation](image)

   **SOLUTION:**

   See the image following part (f).
(b) For each pair \(A\) and \(B\) of vertices of the grid joined by a line, add the line segment joining \(T(A)\) to \(T(B)\) to your plot. This gives a rough picture of what \(T\) is doing.

**SOLUTION:**
See the image following part (f).

(c) What is the image of the \(x\)-axis under \(T\)? The \(y\)-axis?

**SOLUTION:**
The image of the \(x\)-axis is the line \(y = x\). The image of the \(y\)-axis is the line \(y = -x\). To see this, parametrize the \(x\)-axis as \(r(t) = (t, 0), -\infty < t < \infty\). Then \(T(r(t)) = (t, t), -\infty < t < \infty\), which traces out the line \(y = x\). Do the same for the \(y\)-axis.

(d) Consider the line \(L\) given by \(x + y = 1\). What is the image of \(L\) under \(T\)? Is it a circle, an ellipse, a hyperbole, or something else?

**SOLUTION:**
Parametrize \(L\) by \(r(t) = (t, 1 - t), -\infty < t < \infty\). \(T(L)\) is parametrized by \(T(r(t)) = (t - 2(1 - t), t + 2(1 - t)) = (3t - 2, -t + 2)\). These are the parametric equations of a line.

(e) Consider the circle \(C\) given by \(x^2 + y^2 = 1\). What is the image of \(C\) under \(T\)?

**SOLUTION:**
Parametrize \(C\) by \(r(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi\). Then \(T(r(t)) = (\cos t - 2\sin t, \cos t + 2\sin t), 0 \leq t \leq 2\pi\). Note that if we let \(x = \cos t - 2\sin t, y = \cos t + 2\sin t\), then \(y - x = 4\sin t\) and \(y + x = 2\cos t\). So the curve \(T(C)\) satisfies the equation \((\frac{y-x}{4})^2 + (\frac{y+x}{2})^2 = 1\). This is the equation of an ellipse.

(f) Add \(T(L)\), \(T(C)\) and \(T(\ldots)\) to your picture. Check your answer with the instructor.

**SOLUTION:**

![Diagram](image)

**Note:** The transformation \(T\) is a particularly simple sort called a *linear transformation*.

2. Consider the transformation \(T(x, y) = (y, x(1 + y^2))\). Draw the image of the picture below under \(T\).
SOLUTION:
Label the 5 line segments as at left below. The image of the left hand picture is the right hand picture.

We can figure this out as follows. First parametrize the line segments:

\[ r_A(t) = (0, t), 0 \leq t \leq 1 \quad r_B(t) = (t, 0), 0 \leq t \leq 1 \]
\[ r_C(t) = (1, t), 0 \leq t \leq 1 \quad r_D(t) = (t, 1), 0 \leq t \leq 1 \]
\[ r_E(t) = (t, t), 0 \leq t \leq 1 \]

Then compute the image under \( T \) of each of these:

\[ T(r_A(t)) = (t, 0), 0 \leq t \leq 1 \quad T(r_B(t)) = (0, t), 0 \leq t \leq 1 \]
\[ T(r_C(t)) = (t, 1 + t^2), 0 \leq t \leq 1 \quad T(r_D(t)) = (1, 2t), 0 \leq t \leq 1 \]
\[ T(r_E(t)) = (t, t(1 + t^2)), 0 \leq t \leq 1 \]

Graphing each of these gives the image above at left.

3. In this problem, you'll construct a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) which rotates counter-clockwise about the origin by \( \pi/4 \), as shown below.
(a) Give a formula for $T$ in terms of polar coordinates. That is, how does rotation affect $r$ and $\theta$?

SOLUTION:

$$T(r, \theta) = (r, \theta + \pi/4)$$

(b) Write down $T$ in terms of the usual rectangular $(x, y)$ coordinates. Hint: first convert into polar, apply part (a) and then convert back into rectangular coordinates.

SOLUTION:

First convert $(x, y)$ into polar:

$$(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(y/x))$$

Then apply $T$ in polar coordinates:

$$T(r, \theta) = (r, \theta + \pi/4)$$

Then convert the result to rectangular coordinates:

$T(x, y) = (r \cos(\theta + \pi/4), r \sin(\theta + \pi/4))$, where $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$.

Recall the double angle formulas $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$ and $\sin(a + b) = \sin(a) \cos(b) + \sin(b) \cos(a)$. Using these we see that

$$\cos(\theta + \pi/4) = \cos \theta \cos(\pi/4) - \sin \theta \sin(\pi/4) = \sqrt{2}/2 (\cos \theta - \sin \theta)$$

and

$$\sin(\theta + \pi/4) = \sin \theta \cos(\pi/4) + \sin(\pi/4) \cos \theta = \sqrt{2}/2 (\sin \theta + \cos \theta).$$

Hence we have

$$r \cos(\theta + \pi/4) = \sqrt{2}/2 (r \cos \theta - r \sin \theta) = \sqrt{2}/2 (x - y)$$

and

$$r \sin(\theta + \pi/4) = \sqrt{2}/2 (r \sin \theta + r \cos \theta) = \sqrt{2}/2 (x - y).$$

So we have

$$T(x, y) = (\sqrt{2}/2(x - y), \sqrt{2}/2(x - y)).$$