Thursday, October 1 * Solutions * Taylor series, the 2nd derivative test, and changing coordinates.

1. Consider $f(x, y) = 2 \cos x - y^2 + e^{xy}$.

   (a) Show that $(0,0)$ is a critical point for $f$.

   **SOLUTION:**
   \[
   \frac{\partial f}{\partial x}|_{(0,0)} = (-2 \sin x + ye^{xy})|_{(0,0)} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}|_{(0,0)} = (-2y + xe^{xy})|_{(0,0)} = 0
   \]

   (b) Calculate each of $f_{xx}$, $f_{xy}$, $f_{yy}$ at $(0,0)$ and use this to write out the 2nd-order Taylor approximation for $f$ at $(0,0)$.

   **SOLUTION:**
   The second order Taylor approximation of a function $f(x, y)$ at $(0,0)$ is given by
   \[
   T_2(x, y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + (f_{xx}(0,0)/2)x^2 + (f_{xy}(0,0)/2)y^2 + f_{yy}(0,0)xy.
   \]
   For this problem we have $f_{xx} = -2 \cos x + ye^{xy}$, $f_{xy} = 2 + xe^{xy}$, and $f_{yy} = e^{xy} + xye^{xy}$. So $f_{xx}(0,0) = -2 = f_{yy}(0,0)$ and $f_{xy}(0,0) = 1$. Also $f(0,0) = 3$. So the second order Taylor approximation for $f$ at $(0,0)$ is $g(x, y) = 3 - x^2 - y^2 + xy$.

2. Let $g(x, y)$ be the approximation you obtained for $f(x, y)$ near $(0,0)$ in 1(b). It’s not clear from the formula whether $g$, and hence $f$, has a min, max, or a saddle at $(0,0)$. Test along several lines until you are convinced you’ve determined which type it is. In the next problem, you’ll confirm your answer in two ways.

   **SOLUTION:**
   Let’s test a general line $y = mx$ which goes through $(0,0)$ as $x \to 0$. Then $g(x, mx) = 3 - x^2 - m^2x^2 + mx^2 = 3 - (1 - m + m^2)x^2$. The polynomial $1 - m + m^2$ is always positive (it opens upward and has its global minimum at $m = 1/2$ where $1 - m + m^2 > 0$). So $g(x, mx)$ is always a downward opening parabola. This suggests that $(0,0)$ is a relative maximum.

3. Consider alternate coordinates $(u, v)$ on $\mathbb{R}^2$ given by $(x, y) = (u - v, u + v)$.

   (a) Sketch the $u$- and $v$-axes relative to the usual $x$- and $y$-axes, and draw the points whose $(u, v)$-coordinates are: $(-1,2), (1,1), (1,-1)$.

   **SOLUTION:**
   If we express $u$ and $v$ in terms of $x$ and $y$ we get $u = 1/2(x + y)$ and $v = 1/2(y - x)$. So the $u$-axis is given in $x$ and $y$ coordinates by all multiples of the vector $(1,1)$ and the $v$-axis is given by all multiples of the vector $(-1,1)$. The two axes and the points are shown below.
(b) Express \( g \) as a function of \( u \) and \( v \), and expand and simplify the resulting expression.

**SOLUTION:**

\[
3 - x^2 - y^2 + xy = 3 - (u - v)^2 - (u + v)^2 + (u - v)(u + v) = 3 - (u^2 - 2uv + v^2) - (u^2 + 2uv + v^2) + u^2 - v^2 = 3 - u^2 - 3v^2.
\]

(c) Explain why your answer in 3(b) confirms your answer in 2.

**SOLUTION:**

This is an elliptic paraboloid (in \( uv \) coordinates) opening downward with maximum at \((0, 0, 3)\), so it confirms that \((0, 0)\) is a local maximum (\((0, 0)\) goes to \((0, 0)\) under the transformation, so this reasoning makes sense).

(d) Sketch a few level sets for \( g \). What do the level sets of \( f \) look like near \((0, 0)\)?

**SOLUTION:** The level sets are sketched for \( g = 2.7, 2.8, 2.9 \) on the left and for \( f = 2.7, 2.8, 2.9 \) on the right. The level sets for \( g \) are ellipses that approximate the level sets of \( f \) close to \((0, 0)\). The ellipses shrink as they get closer to \( g(x, y) = 3 \), which consists of the single solution \((x, y) = (0, 0)\).

(e) It turns out that there is always a similar change of coordinates so that the Taylor series of a function \( f \) which has a critical point at \((0, 0)\) looks like \( f(u, v) \approx f(0, 0) + au^2 + bv^2 \). In fact this is why the 2\(^{nd}\) derivative test works.

Double check your answer in 2 by applying the 2\(^{nd}\)-derivative test directly to \( f \).

**SOLUTION:**
The Hessian $f_{xx}f_{yy} - (f_{xy})^2$ is $(-2)(-2) - 1^2 = 3 > 0$ at $(0,0)$ and $f_{xx}(0,0) = -2 < 0$. So $f$ has a relative maximum at $(0,0)$ as suspected.

4. Consider the function $f(x, y) = 3xe^y - x^3 - e^{3y}$.

(a) Check that $f$ has only one critical point, which is a local maximum.

**SOLUTION:**

$f_x = 3e^y - 3x^2$ and $f_y = 3xe^y - 3e^{3y}$. $f_y = 0$ only if $x = e^{2y}$ and $f_x = 0$ only if $e^y = x^2$. Solving these simultaneously we see that $x$ must satisfy $(x^2)^2 = (e^y)^2 = x$, so $x = 0, -1, \text{ or } 1$. But $x = e^{2y} > 0$ so the only critical point is $x = 1, y = 0$. Calculating, we see that $f_{xx}(1,0) = f_{yy}(1,0) = -6$ and $f_{xy}(1,0) = 3$. So the Hessian $f_{xx}f_{yy} - (f_{xy})^2 = 36 - 9 = 27 > 0$ at $(1,0)$. Since $f_{xx}(1,0) < 0$, the second derivative test tells us that $f(1,0) = 1$ is a local maximum.

(b) Does $f$ have an absolute maxima? Why or why not?

**SOLUTION:**

$f$ does not have an absolute maximum. For instance if we take the trace curve $y = 0$ we get $f(x,0) = 3x - x^3 - 1$, which is unbounded as $x \to \infty$. Absolute maxima and minima are only guaranteed over a closed and bounded set in the domain. The plane $\mathbb{R}^2$ is closed but not bounded, so there is no guarantee that a continuous function will achieve an absolute maximum or minimum over $\mathbb{R}^2$. 