Do not open this test booklet until you are told to do so.

Turn off all electronic devices and put away all items except a pen/pencil and an eraser.

No baseball caps, hoodies, etc. or dark sunglasses. All hats are to be removed.

There is no partial credit on multiple-choice questions. For all other questions, you must show sufficient work to justify your answer.

While the test is in progress, we will not answer questions concerning the test material.

If you finish early, quietly and respectfully get up and hand in your exam.

When time is up, you will be instructed to put down your writing utensil, close your exam and remain seated. Anyone seen continuing to write after time is called will have their exam marked and lose all points on the page they are writing on.

Good luck. You have **60 minutes** to complete this exam.
1. (7 points) Suppose that the price $p$ (in dollars) and the weekly sales of $x$ items satisfy the demand equation $2p^3 + x^2 = 4500$. Determine the rate at which sales are changing at a time when $x = 50$, $p = 10$, and the price is falling at a rate of $\$1$ per week. Simplify your answer.

$$2p^3 + x^2 = 4500$$

Take derivative w.r.t. time,

$$\frac{d}{dt} (2p^3 + x^2) = \frac{d}{dt} 4500$$

$$6p^2 \frac{dp}{dt} + 2x \frac{dx}{dt} = 0$$

Plug in values,

$$6(10)^2(-1) + 2(50) \frac{dx}{dt} = 0$$

$$-600 + 100 \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{600}{100}$$

$$\frac{dx}{dt} = 6$$

2. (7 points) Use implicit differentiation to find the slope of the tangent line to the given curve at the point $(x, y) = (2, -1)$.

$$x^3 + y^3 = 1 + 3xy^2$$

**Solution**

$$\frac{d}{dx} (x^3 + y^3) = \frac{d}{dx} (1 + 3xy^2)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3y^2 + 6xy \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - 6xy \frac{dy}{dx} = 3y^2 - 3x^2$$

$$(3y^2 - 6xy) \frac{dy}{dx} = 3y^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 3x^2}{3y^2 - 6xy}$$

Slope of the tangent line at $(x, y) = (2, -1)$ is;

$$\frac{dy}{dx} = \frac{3(-1)^2 - 3(2)^2}{3(-1)^2 - 6(2)(-1)}$$

$$\frac{dy}{dx} = \frac{-9}{15}$$

$$\frac{dy}{dx} = -\frac{3}{5}$$
3. (10 points) A manufacturer estimates that the marginal cost of producing \( q \) units of a certain commodity is \( C'(q) = 3q^2 - 10q + 20 \) dollars per unit. If the cost of producing 1 unit is $20, what is the cost of producing 3 units?

Solution

\[
\int C'(q) \, dq = \int (3q^2 - 10q + 20) \, dq
\]

\[
C(q) = q^3 - 5q^2 + 20q + c_1
\]

Now using \( C(1) = 20 \), we get

\[
C(1) = 1^3 - 5(1)^2 + 20(1) + c_1 = 20
\]

\[
20 = 1 - 5 + 20 + c_1 \implies c_1 = 4
\]

\[
C(q) = q^3 - 5q^2 + 20q + 4
\]

\[
C(3) = (3)^3 - 5(3)^2 + 20(3) + 4
\]

\[
C(3) = 27 - 45 + 60 + 4 = 46
\]

\[
C(3) = $46
\]

4. (10 points) Evaluate the following indefinite integrals.

(a) \[
\int \left( y^3 + \frac{1}{y} + e^{2y} \right) \, dy = \frac{y^4}{4} + \ln(|y|) + \frac{e^{2y}}{2} + C.
\]

(b) \[
\int x^3 \sqrt{x^2 + 1} \, dx.
\]

Solution

\[
u = x^2 + 1 \implies du = 2x \, dx
\]

\[
\int x^3 \sqrt{x^2 + 1} \, dx = \frac{1}{2} \int (u - 1) \sqrt{u} \, du = \frac{1}{2} \int (u^{3/2} - u^{1/2}) \, du = \frac{1}{2} \left( \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right) + c
\]

\[
\int x^3 \sqrt{x^2 + 1} \, dx = \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C
\]
5. (10 points) Given the revenue and cost functions \( R = 50x - \frac{2}{5}x^2 \) and \( C = 5x + 15 \) (in dollars), where \( x \) is the daily production (and sales), find the following when 40 units are produced daily and the rate of change of production is 10 units per day.

(a) The rate of change of revenue with respect to time.

Solution:

\[
\frac{dR}{dt} = 50 \frac{dx}{dt} - \frac{4}{5} x \frac{dx}{dt}
\]

\[
\frac{dR}{dt} = 50(10) - \frac{4}{5}(40)(10)
\]

\[
\frac{dR}{dt} = 500 - 320
\]

\[
\frac{dR}{dt} = 180
\]
Revenue is increasing at the rate of $180 per day.

(b) The rate of change of cost with respect to time.

Solution:

\[
\frac{dC}{dt} = 5 \frac{dx}{dt}
\]

\[
\frac{dC}{dt} = 5(10) = 50
\]
Cost is increasing at the rate of $50 per day.

(c) The rate of change of profit with respect to time.

Solution:

Profit = Revenue - Cost
\[
P(t) = R(t) - C(t)
\]

\[
\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt}
\]

\[
\frac{dP}{dt} = 180 - 50
\]

\[
\frac{dP}{dt} = 130
\]
Profit is increasing at the rate of $130 per day.
6. (10 points) A rectangular container with open top is required to have a volume of 16 cubic meters. Also, one side of the rectangular base is required to be 4 meters long. If material for the base costs $8 per square meter, and material for the sides costs $2 per square meter, find the dimensions of the container so that the cost of material to make it will be a minimum. (Hint: Volume of a box with sides of length $l$, $w$, and $h$ is $lwh$.)

Solution:

We want to minimize the cost $C = 8(4w) + 2(2wh) + 2(8h) = 32w + 4wh + 16h$

subject to $4wh = 16 \implies h = \frac{4}{w}$

$C = 32w + 4w\left(\frac{4}{w}\right) + 16\left(\frac{4}{w}\right)$

$C(w) = 32w + 16 + \frac{64}{w}$ \hspace{1cm} (0, 4)

$C'(w) = 32 - \frac{64}{w^2}$

$C'(w) = 0 \implies w^2 = 2 \implies w = \sqrt{2}$ belongs to (0, 4).

$C''(w) = \frac{128}{w^3} \implies C''(\sqrt{2}) > 0 \implies w = \sqrt{2}$ is a local minimum.

Using one point critical theorem, $C$ is minimum when $w = \sqrt{2}$ and $h = \frac{4}{\sqrt{2}}$.
7. (8 points) You do not need to show work for this question. Just circle the best answer for each part. (2 points each)

(a) Using linear approximation formula, the approximate value of $\sqrt{35}$ is:

(A) $\frac{73}{12}$  
(B) $\frac{71}{12}$  
(C) $\frac{75}{12}$  
(D) $\frac{70}{12}$  
(E) $\frac{72}{12}$

Solution: B

$f(x) = \sqrt{x}$, $x = 36$, $dx = -1$

$f(x + dx) \approx f(x) + f'(x) \, dx$

$f(36 - 1) \approx f(36) + \frac{1}{2(\sqrt{36})}(-1)$

$f(35) \approx \sqrt{36} - \frac{1}{12}$

$\sqrt{35} \approx 6 - \frac{1}{12} = \frac{72}{12} - \frac{1}{12} = \frac{71}{12}$

(b) The absolute minimum value of $f(x) = 2x^3 - 3x^2 + 3$ on the interval $[-1, 2]$ is:

(A) $-2$  
(B) $3$  
(C) $2$  
(D) $7$  
(E) $9$

Solution: A

$f'(x) = 6x^2 - 6x = 0 \implies x = 0, x = 1$

$f(0) = 3, f(1) = 2, f(-1) = -2, f(2) = 7$

(c) A boat leaves a given point and travels north at 8 mph. Another boat leaves the same point at the same time and travels east at 6 mph. At what rate is the distance between the two boats changing at the instance the boats have traveled for 1 hour?

(A) 2 mph  
(B) 4 mph  
(C) 7 mph  
(D) 10 mph  
(E) 12 mph

Solution: D

$\frac{dz}{dt} (x^2(t) + y^2(t)) = \frac{dz}{dt} z^2(t)$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$x \frac{dx}{dt} + y \frac{dy}{dt} = z \frac{dz}{dt}$

$6(6) + 8(8) = 10 \frac{dz}{dt} \implies \frac{dz}{dt} = 10$

(d) A worker is cutting a square from a piece of metal. The specifications call for an area of 25 $ft^2$ with an error of no more than $\frac{1}{5} \, ft^2$. How much error can be tolerated in the length of each side of the square to ensure the area is within tolerance? (Note: The area of a square of side length $x$ is $x^2$.)

(A) $\frac{1}{50}$  
(B) $\frac{1}{10}$  
(C) $\frac{1}{70}$  
(D) $\frac{1}{30}$  
(E) $\frac{1}{20}$

Solution: A