TALK NOTES: QUANTUM MECHANICS OF A PARTICLE ON A GROUP

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Here I give my notes detailing section 2 of Krzysztof Gawedski's lecture notes: "Conformal Field Theory: a case study." Much of the material for the Peter-Weyl Theorem comes from Terry Tao's blog post on the subject, and my many discussions with Bob Hingtgen at UCSC. Any typos should be assumed to be my own.

I think it's good to have a goal or big idea for a talk. So I'll just begin with that: this quantization process Tom's already alluded to is a fairly widespread phenomenon, and if you can leverage a lot of symmetry from the 'classical picture', you'll be able to do as much as you'd like in the 'quantum picture.'

1. Classical Dynamics on G

We've (presumably) already heard about the Classical Action for our physical system. In general, this is a scalar-valued functional defined on maps in our physical system. In this section of Gawedski we consider the action coming from the length functional. In this section, the underlying "space" we'll be considering will G, and the physical system will be some open dense subset $\mathcal{P} \subset T^*G$, where this is the "phase space" of our

Fix our compact Lie group G, and a riemannian metric γ on G. Then we'll define

$$S(x) = \frac{1}{2}\int_0^T \gamma(\dot{x}(t), \dot{x}(t))dt = \frac{1}{2}\int_0^T \gamma_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}dt,$$

for all paths in the group $x : [0, T] \to G$. This S is the classical action coming from the "energy" of a path (we shall see that the minimal paths with respect to this action are precisely reparamatrizations of geodesics). The classical solutions $\delta S = 0$ of this action are solutions to the associated Euler-Lagrange equations

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0.$$

These correspond to the Geodesic equations on G for this action, and this choice of metric γ. So, which metric should we consider?

Since we're considering a compact Lie Group a natural such choice comes from considering a bi-invariant metric, i.e. one invariant under both the left and right translation actions of G acting on itself. Here, this is equivalent to choosing positive, ad-invariant, bilinear form on $\mathfrak{g}$, and then defining it on the other fibers of TG by translating a given tangent space back to the Lie algebra. We shall consider

$$\langle X, Y\rangle = \frac{k}{2}\mathrm{Tr}(X^\intercal Y)$$

where we are implicitly making a choice of embedding G as a subgroup into some matrix group, which is always possible since G is a compact Lie group. This set-up now allows us to introduce our first object of physical interest, the space $\mathcal{P} \subset T^*G$ is the open dense subset of all of phase space which is comprised of classical solutions of our geodesic equation.

Further, on a compact Lie group the exponential map

$$e^{it(-)} : \mathfrak{g} \to G^0,$$

coincides with the Riemannian exponential map, sending a tangent vector to the geodesic with that initial velocity. By Hopf-Rinow, we may connect any point in the group to a geodesic beginning at the identity element. In particular, we write any solution of the geodesic equation in the form

$$g(t) = g_l e^{it\lambda/k} g_r^{-1}$$

for some $g_l, g_r \in G$ and λ is some element of the cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. In this set-up, $g(0) = g_l g_r^{-1}$

This allows us to define the hamiltonian function

$$h : \mathcal{P} \subset T^*G \to \mathbb{R},$$

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as follows,

$$h : (g, p) \mapsto g(t) = g_l e^{it\lambda/k} g_l^{-1} \mapsto \frac{1}{4k}\mathrm{Tr}(\lambda^2) \in \mathbb{R}$$

where λ is the unique initial velocity such that the geodesic beginning at e which connects to g at time 1. This coincides with the classical definition of the hamiltonian as the norm of the momentum: since we define $p(t) = \frac{k}{2i}\frac{d}{dt}g(t)$, then $h := \frac{1}{k}\mathrm{Tr}(p^2) \equiv \frac{1}{k}||p||^2$. (**REMARK**, mention something about this being the principal symbol of the laplacian).

At this point we should mention that $\mathcal{P} \subset T^*G$ really is a symplectic manifold: there is a closed, non-degenerate differential 2-form $\omega \in \Omega^2(T^*G)$ which can be defined by

$$\omega = d\underbrace{\mathrm{tr}(pg^{-1}dg)}_{\alpha}.$$

Let's unpack this. T^*G isn't just any symplectic manifold, it's the cotangent bundle of a manifold. As such it has a canonical symplectic structure given by the differential of the Liouville form $\alpha \in \Omega^1(T^*G)$. This 1-form is a function $\alpha : T(T^*G) \to \mathbb{R}$, but the tangent bundle of the cotangent bundle splits,

$$T(T^*G) \simeq TG \oplus T^*G,$$

(it doesn't split naturally, but we made a choice of metric on G in the first page and this fixes a choice of splitting) and we can use this splitting to define

$$\alpha : TG \oplus T^*G \to \mathbb{R}, \qquad (v, p) \mapsto p(v).$$

So the differential of this 1-form is our symplectic form. One thing this symplectic structure is good for is converting between functions on $\mathcal{P}$ and vector fields on $\mathcal{P}$. We may do this as follows: given a function $f : \mathcal{P} \to \mathbb{R}$, we define it's Hamiltonian vector field X_f by

$$-df(Y) = \omega(X_f, Y) \qquad \forall Y \in \mathfrak{X}(\mathcal{P}),$$

which well-defined by non-degeneracy of ω.

Great, we have a symplectic structure. Back to representation theory. There are two commuting actions of G on $\mathcal{P}$,

$$\text{the left action: } g(t) \mapsto g_0 g(t)$$

$$\text{the right action: } g(t) \mapsto g(t)g_0^{-1}.$$

These both preserve the symplectic structure and the hamiltonian h, because both are defined in terms of our riemannian metric, which is itself bi-invariant. So we should study how this invariance relates to the group structure as this should give representation-theoretic content. Consider the following two functions, defined for each element $t^a \in \mathfrak{g}$, in some basis,

$$\text{the 'left action', } j^a(g) = (1/2)\mathrm{tr}(t^a g_l \lambda g_l^{-1})$$

$$\text{the 'right action', } \tilde{j}^a(g) = (1/2)\mathrm{tr}(t^a g_r \lambda g_r^{-1}).$$

The quotes are from the following correspondences: the hamiltonian vector fields of these functions are the infinitesimal generators for the left (correspondingly right) actions by the exponentials of those basis elements. In symbols

$$j^a \mapsto X_{j^a}, \qquad X_{j^a}(g(t)) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} e^{i\epsilon t^a} \cdot g(t),$$

and similarly for $\tilde{j}^a$. If we picked an orthonormal basis for the $\{t^a\} \subset \mathfrak{g}$ then we have the following

$$h = \frac{2}{k}\sum_a j^a j^a = \frac{2}{k}\sum_a \tilde{j}^a \tilde{j}^a.$$

Alright, now let's quantize this stuff.

2. Quantum Dynamics on G and Peter-Weyl

It's somewhat axiomatic in mathematical physics that classical physics should correspond to symplectic manifolds and real-valued functions on them, as quantum physics should correspond to hilbert spaces and self-adjoint operators on them.

In this case, we'll view a version of this "quantization" procedure as follows: we'll pass from the classical object $\mathcal{P} \subset T^*G$ to a quantum object $L^2(G, dg)$, and try and carry as much of the representation theoretic data as possible. Here we've defined $L^2(G, dg)$ via the normalized Haar measure on G, (really just the Riemannian volume form with respect to the bi-invariant metric we've used throughout).

The quantizations of the left and right actions of G on $\mathcal{P}$ becomes the left and right regular representations of G on $L^2(G)$:

$$f(g) \mapsto f(u^{-1}g) \qquad f(g) \mapsto f(gu),$$

which are unitary, irreducible representations. Further, these give rise to the actions

$$J^a f(g) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f(e^{-i\epsilon t^a} \cdot g), \qquad \widetilde{J}^a f(g) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f(g \cdot e^{i\epsilon t^a}).$$

As we saw from Tom's talk, we saw symmetries of our Lagrangian (like the bi-invariant metric on G creating this property for S) should induce conserved quantities of our system, in particular conservation equations. In this case our conserved quantity was the energy was level sets of the function $h = \frac{1}{4k}\mathrm{Tr}(p^2) = \frac{2}{k} j^a j^a$. The corresponding quantum object is

$$H = \frac{2}{k}\sum_a J^a J^a = \frac{2}{k}\sum_a \widetilde{J}^a \widetilde{J}^a = -\frac{2}{k}\Delta_G,$$

a rescaling of the Laplace-Beltrami operator with respect to our metric on G.

Now, let's try and decompose $L^2(G)$ into simpler, finite-dimensional pieces. As a 'quantum' object, the energy of our hamiltonian H should come in discrete pieces: this corresponds to the spectrum of H, and we obtain a corresponding eigenfunction decomposition,

$$L^2(G, dg) \simeq \bigoplus_{\lambda_j \in \mathrm{Spec}(H)} E_{\lambda_j}^{\oplus \mu(\lambda_j)}$$

where $\mathrm{Spec}(H)$ denotes the distinct eigenvalues (in ascending order), E_{λ_j} denotes the λ_j-eigenspace, and $\mu(\lambda_j)$ the multiplicity of that distinct eigenvalue.

The simplest example of such an eigenfunction decomposition comes from Fourier theory on S^1:

$$L^2(S^1, d\theta) \simeq \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \cdot \{e^{2\pi i k\theta}\},$$

but this is better than an eigenfunction expansion. The Fourier basis is given by irreducible representations of S^1 in $L^2(S^1)$, given by the functions $\{e^{2\pi i k\theta}\}_{k\in\mathbb{Z}}$, (i.e. representations for which there is no proper subrepresentation closed the image of the rep). This theorem also implies that (i) the irreps are pairwise orthogonal with respect to the Hilbert space inner product, and (ii) finite linear combos of these irreps are dense in $L^2(S^1)$. This structure theorem for S^1 holds much more broadly for other compact Lie groups.

Since this group was abelian, every representation must be 1-dimensional. So for more general compact groups, the analogous statement is given by

$$L^2(G, dg) \simeq \bigoplus_{[R, V_R]} V_R \otimes V_{\overline{R}},$$

where we take this direct sum of equivalence classes of irreducible representations $R : G \to GL(V_R)$ satisfying $\overline{R}$ is the complex conjugate to R, $g_{\overline{R}} = \overline{g_R}$, and $V_R^* = \overline{V}_R$. We now also have a nice way of representating elements in these finite pieces, with respect to the spectral decomposition: every representation space $V_R \equiv E_{\lambda_j}$ is isomorphic to a given eigenspace of H

$$\text{with an eigenvalue } \lambda_j = \mathrm{Tr}_{V_R}(R(t^a t^a)) =: \mathrm{Tr}_{V_R}(H)$$

A particular class of functions $\phi : G \to \mathbb{C}$ in $L^2(G)$ are the *matrix coefficients*, of the following form. Consider $R : G \to GL(V_R)$ is a unitary irreducible representation, if $\{e_i\} \subset V_R$ is an orthonormal basis, and let (g_R^{ij}) be the unitary matrix defined by $R(g) = g_R \in GL(V_R)$. So a matrix coefficient is a function $\phi(g) = g_R^{kl}$, for some k, l fixed.

(Equivalently, this can be defined in a basis-independent way as follows:

$$\phi(g) = \ell(R(g)v) = \ell(g_R(v))$$

and for some fixed $v \in V$ and $\ell \in V^*$.)

Writing the matrix coefficients of a representation in terms of the eigenfunction basis for that eigenspace/representation space, we have the following correspondence

$$E_{\lambda_R} \otimes E^*_{\lambda_R} = V_R \otimes V_{\overline{R}} \ni e^i_R \otimes \overline{e^j_R} \quad \mapsto \quad d_R^{-\frac{1}{2}} \overline{g^{ij}_R} \in L^2(G, dg)$$

That these matrix coefficients are preserve the L^2 inner-product is a consequence of the Schur orthogonality relations, given below:

Lemma 2.1. ***Schur's Lemma***

Let $R : G \to U(H)$ and $R' : G \to U(H')$ be unitary irreducible representations, and $T : H \to H'$ a linear map, equivariant with respect to these representations. Then T is zero, or equal to a constant times an isomorphism. In particular, if $R \not\equiv R'$, then there are no non-trivial equivariant maps between H and H'.

Proof. Since the adjoint map $T^* : H' \to H$ is also equivariant, hence so is $T^*T : H \to H$. As a self-adjoint operator, we can apply the spectral theorem to it. Since any closed subspace of T^*T is G-equivariant, this subspace must be either $\{0\}$ or H. Hence, by the spectral theorem T^*T is a multiple of the identity. Similarly for TT^*. Hence T is zero, or a constant multiple of the identity as claimed. □

Corollary 2.2. Let $R : G \to GL(V_R)$ be an irreducible unitary representation, and $T : V_R \to V_R$ a linear map. Then

$$\int_G \overline{g_R} \circ T(-) \circ g_R dg = \frac{\mathrm{Tr}(T)}{d_R} \cdot \mathrm{Id}_{V_R}(-)$$

where $\dim(V_R) = d_R$.

Proof. The left hand side is equivariant, hence it must be a constant times the identity. Taking the trace of both sides, we see the trace of the left-hand side coincides with the right hand side. □

Using Schur's Lemma, and choosing T to be the identity immediately gives the following:

Lemma 2.3. ***Schur Orthogonality***

Let $R' : G \to GL(V_R)$ and $R' : G \to GL(V_{R'})$ be unitary irreducible representations. Then we have

$$\int_G \overline{g^{ij}_{R'}} g^{rs}_R dg = \frac{1}{d_R} \delta_{R'R} \delta^{ir} \delta^{js}$$

Proof. Combine the "in particular" portion of Schurs lemma gives, and the previous corollary with a choice of $T = \mathrm{Id}_{V_R}$ □

The fact that these matrix coefficients span a dense subset of $L^2(G, dg)$ is a result of denseness of the Laplace eigenfunctions, and that

$$H(f(u^{-1}g)) = (Hf)(u^{-1}g)$$

i.e. the left/right regular representations of G on $L^2(G, dg)$ commutes with the Laplacian (since these actions are Hilbert space isometries from our choice of Haar measure).

Now we can state the analog of the complex exponentials on the circle: the characters $\chi_R := \mathrm{Tr}_{V_R}(g_R)$, which are class functions, i.e. those constant on the conjugacy classes

$$\mathcal{C}_\lambda = \{g_0 e^{2\pi i \lambda / k} g_0^{-1} | g_0 \in G\}$$

for λ in the Cartan $\mathfrak{t} \subset \mathfrak{g}$. Recall that the classical space $\mathcal{P}$ is parametrized by these conjugacy classes. Note that the duals of these functions are precisely the diagonals of the representation in the eigenfunction bases,

$$\overline{\chi}_R = \sum_{i=1}^{d_R} d_R^{-\frac{1}{2}} e^i_R \otimes \overline{e^i_R},$$

(I'm not sure how to interpret this fact. I think this means the eigenfunctions of the quantum hamiltonian H is constant on the level sets of the classical hamiltonian).

3. The Heat Propagator and Path Integral

The symmetries of the lagrangian in the classical action mentioned in section 1 implies a conservation law. Namely conservation of energy. The equation defining this conservation may be put in divergence form $\partial_t f + \operatorname{div}(X_f) = 0$. Using the symplectic structure, and our choice of riemannian metric this is equivalent to the heat equation

$$\begin{cases} (\partial_t + H)f(g,t) = 0 \\ f|_{t=0} = f(g), \end{cases}$$

so we might as well introduce the fundamental solution of this PDE, the Heat Propagator! This is the operator $e^{-tH} : L^2(G) \to L^2(G)$ satisfying

$$f(x) \mapsto \begin{cases} (\partial_t + H)e^{-tH}f(g) = 0 \\ \lim_{t\to 0} e^{-tH}f(g) = f(g) \end{cases}$$

sending an "initial temperature distribution" to the corresponding solution of the heat equation. We can also obtain it as an integral operator with integral kernel

$$e^{-tH}f(g) = \int_G e(t, g_0, g_1)f(g_1)dg_1,$$

$$e(t, g_0, g_1) = \sum_R d_R e^{-\frac{2}{k}t\lambda_R}\chi_R(g_0 g_1^{-1}) = \sum_{\lambda_j \in \operatorname{Spec}(H)} e^{-t\lambda_j} e_j(g_0) \otimes \overline{e_j(g_1)}.$$

This representation, and the all-crucial localization property of the heat kernel (this operator acts like the identity as $t \to 0$) immediately gives a decomposition for a δ mass by setting $t = 0$,

$$\delta_{g_0}(g_1) \sum_{\lambda_R} d_R \chi_R(g_0 g_1^{-1}),$$

and similarly delta masses on conjugacy classes

$$\delta_{\mathcal{C}_\lambda}(g_1) = \int_G \delta_{g_0 e^{2\pi i\lambda/k} g_0^{-1}}(g_1)dg_0 = \sum_R \chi_R(e^{2\pi i\lambda/k})\overline{\chi_R(g_1)}.$$

Combining these facts, we can decompose the Haar measure into a product measure defined on the conjugacy classes

$$dg = \frac{1}{|T|}\left|\prod_{\lambda\in\mathfrak{t}}(e^{2\pi i\lambda/k})\right|^2 \delta_{\mathcal{C}_\lambda}(g)d\lambda dg$$

where $d\lambda$ is the pushforward of the Haar measure under the on the Cartan subgroup $T \subset G$ under the exponential map. So for class functions, which are constant on conjugacy classes, we obtain the Weyl formula

$$\int_G f(g)dg = \int_T f(\lambda)\left|\prod_{\lambda\in\mathfrak{t}}(e^{2\pi i\lambda/k})\right|^2 d\lambda.$$

I mention this now because we'll see another version of this Weyl Formula in Josh's talk tomorrow, and he'll be annoyed if I don't mention it.

Now we can mention how this all relates to the path-integral Tom mentioned. He said that this was discovered by Feynman, and the first version of the following theorem was proven by Kac

Theorem 3.1. ***Feynman-Kac Formula***

If we define $Dg = dW_{g_0,g_1}(g)$ to be the Brownian bridge measure, supported on continuous paths, then we have the following

$$e^{-tH}f(g_0) = \int_{\substack{g:[0,t]\to G \\ g(0)=g_0, \\ g(t)=e^{-tH}f(g_0)}} f(g(t))e^{-S(g)}Dg$$

This also defines the Thermal correlation function

$$\text{"Correlation functions of these representations"} \quad \left\langle \prod_{n=1}^{N} g_{R_n}^{i_n j_n}(t_n) \right\rangle_t = \frac{\int \prod_{n=1}^{N} g_{R_n}^{i_n j_n} e^{-S(g)}Dg}{\int e^{-S(g)}Dg}$$

where we're integrating again over paths $g : [0, t] \to G$. There insertion points should be thought of in terms of the picture Tom described earlier

$$t_0 - - - t_1 - - - \ldots - - - t_n$$

after we've fixed an ordering on the points in the interval. With this choice of ordering, and the Feynman-Kac formula, we can compute this via Harmonic analysis on G, i.e. hilbert space traces.

$$\mathrm{Tr}(e^{-tH}) = \sum_R e^{-t\lambda_R} = \int_{\mathrm{diag}_G} e(t, g_0, g_0) dg_0$$

This is all computable since the heat kernel is a trace-class operator (in fact, it's a compact operator: every trace class operator on a Hilbert space is a limit of finite rank operators. We obtained the heat kernel as a limit of finite sums of projections onto eigenspaces). So we compute the right-hand side via

$$\left\langle \prod_{n=1}^{N} g_{R_n}^{i_n j_n}(t_n) \right\rangle_t = \frac{\mathrm{Tr}(e^{-t_1 H} \circ g_{R_1}^{i_1 j_1} \circ e^{-(t_2 - t_1)H} g_{R_2}^{i_2 j_2} \circ \ldots \circ e^{-(t - t_N)H} g_{R_N}^{i_N j_N})}{\mathrm{Tr}(e^{-tH})}$$

But by the Schur orthogonality relations, the right hand side is computable just in terms of the matrix coefficients (with respect to the eigenbasis):

$$(e_{R_1}^{i_1 j_1} \otimes \overline{e_{R_1}^{i_1 j_1}}, g_R^{ij} e_{R_2}^{i_2 j_2} \otimes e_{R_2}^{i_2 j_2}) = (d_{R_1} d_{R_2})^{1/2} \int_G \overline{g_{R_1}^{i_1 j_1}} g_R^{ij} g_{R_2}^{i_2 j_2} dg$$

which is much more representation theoretic: we have a ring structure on these irreducible representation spaces, and these correlations functions may now be viewed as giving the coefficients in the decomposition of tensor products into irreducible representations

$$V_R \otimes V_{R_1} = \bigoplus_{R_2} M_{R_1 R}^{R_2} \otimes V_{R_2}.$$

where we can compute the dimensions of the multiplicity spaces via

$$\dim(M_{R_1 R}^{R_2}) \equiv N_{R_1 R}^{R_2} = \int_G \overline{\chi_{R_2}(g)} \chi_R(g) \chi_{R_1}(g) dg.$$

So these dimensions are like the structure constants of our ring.

Further, we have a near identical correlation function for matrix coefficients localized on conjugacy classes:

$$\left\langle \prod_{n=1}^{N} g_{R_n}^{i_n j_n}(t_n) \right\rangle_{t, \lambda_1, \lambda_2} = \frac{\int \prod_{n=1}^{N} g_{R_n}^{i_n j_n} \delta_{c_{\lambda_1}}(g(0)) \delta_{c_{\lambda_2}}(g(t)) e^{-S(g)} Dg}{\int \delta_{c_{\lambda_1}}(g(0)) \delta_{c_{\lambda_2}}(g(t)) e^{-S(g)} Dg}$$