Conservative Vector Fields in $\mathbb{R}^3$

**Def.:** $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is **conservative** if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$

**Ex:** $\vec{F} = (x, y), \quad F = \frac{1}{2}(x^2 + y^2), \quad \nabla F = (x, y) \quad \checkmark$

$\vec{F} = (-y, x), \quad \times \quad b/c \quad \frac{\partial x}{\partial x} = 1 \neq -1 = \frac{\partial (-y)}{\partial y}$

**Thm 1** $\vec{F}$ on $D$ in $\mathbb{R}^n$ is conservative if and only if

$$\int_C \vec{F} \cdot d\vec{r} = 0 \quad \text{for every closed curve } C \text{ in } D$$

**Thm 2** If $D$ in $\mathbb{R}^2$ is simply connected (no holes), then $\vec{F} = (P, Q)$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

**Pf (Thm 2)** Idea: use Green's Thm & Thm 1.

If $C$ in $D$ is closed, then since $D$ has no holes, $C$ is the boundary of some region $R$, then Green's Thm says

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iiint_R 0 \, dA = 0.$$  

Then Thm 1 $\Rightarrow \vec{F}$ is conservative.

- What about $\mathbb{R}^3$, any form like Thm 2 for $\mathbb{R}^3$?

Suppose $\vec{F} = \nabla f = (F_x, F_y, F_z)$ is conservative

$$\Rightarrow \text{curl } \vec{F} = \text{curl}(\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial y \partial z} \end{pmatrix} = \mathbf{0}.$$
Ex) \( \mathbf{F} = (y, z, x) \) is NOT conservative, b/c
\[ \text{Curl } \mathbf{F} = (1, -1, -1) \neq \mathbf{0}. \]

Thm 3] \( \mathbf{F} \) on all of \( \mathbb{R}^3 \) is conservative if and only if
\[ \text{Curl } \mathbf{F} = \mathbf{0} \, \text{ everywhere}. \]

Reason If \( \text{Curl } \mathbf{F} = \mathbf{0} \) and \( C \) is a closed curve in \( \mathbb{R}^3 \),
\( C \) is the boundary of some (orientable)
Surface \( S : \ C = \partial S \).

Stokes Thm \( \Rightarrow \)
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{Curl } \mathbf{F}) \cdot \mathbf{n} \, dA \]
\[ = \iint_S \mathbf{0} \cdot \mathbf{n} \, dA = 0 \]
\( \Rightarrow \) Thm 1 \( \Rightarrow \) conservative.