Flux in \( \mathbb{R}^2 \)

\[ \text{Curve } C \]

\[ \text{River } \mathbb{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ Fluid } \]

**Q** What is the rate that fluid is crossing \( C \)? (the flux)

**Unit:** \( \text{Area} \div \text{time} \)

Let \( \vec{n} \) be a unit normal vector field for \( C \).

\[ \text{Flux} = \int_C (\vec{F} \cdot \vec{n}) \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \| \vec{r}'(t) \| dt \]

**Reason:**

- Look at small (enough) part of a curve, we can consider \( \vec{r} \) is constant.

- Area of fluid crossing segment

\[ A \left( \vec{F} \cdot \vec{n} \right) \| \vec{r}'(t) \| dt \]

\[ \| \vec{r}'(t) \| dt = ds \text{ as } dt \rightarrow 0 \]

**Remark**

- Before (work done by \( \vec{F} \) along \( C \))

  we integrate \( \vec{F} \) ALONG \( C \)

- How we integrate \( \vec{F} \) ACROSS \( C \)

- To compute \( \vec{n}(t) \) (unit vector!)

  given \( \vec{r}(t) = (x_1(t), x_2(t)) \)

  \[ \Rightarrow \vec{n}(t) = \left( \frac{x_2'(t)}{\| \vec{r}'(t) \|}, -\frac{x_1'(t)}{\| \vec{r}'(t) \|} \right) \]
Given $\vec{F} = (F_1, F_2)$, closed curve $C$.

$$\Phi = \oint_C (\vec{F} \cdot \vec{n}) \, ds = \int_a^b \left( F_1(\vec{r}(t)), F_2(\vec{r}(t)) \right) \cdot \left( \vec{r}'(t), -\vec{r}'(t) \right) \, dt$$

$$= \int_a^b \left( F_1(\vec{r}(t)) \vec{r}'(t) - F_2(\vec{r}(t)) \cdot \vec{r}'(t) \right) \, dt$$

$$= \int_C \vec{G} \cdot d\vec{s} \quad \text{if } \vec{G} = (-F_2, F_1)$$

$$= \iint_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \, dA \quad \text{by Green's Thm}$$

**Def. Divergence:** $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \vec{F} = (F_1, F_2)$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad \text{it is a number}$$

Where $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ (as a vector)

(Recall $\nabla$ think of it as scalar product $\vec{v}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$)

**Meaning:** $\text{div } \vec{F} =$ rate of expansion of area under the flow indicated by $\vec{F}$

**Ex.** $\vec{F} = (x, y)$

Dye the water

How fast is color expanding?
Divergence Thm \((\mathbb{R}^2)\)

\[
\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{D} \text{div} \mathbf{F} \, dA
\]

meaning After time \(\Delta t\), the fluid in \(D\) now fills the region \(D'\)

\[
\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1 + \Delta t \cdot r
\]

\[
r = \text{ave. rate of expansion} = \frac{1}{\text{Area}(D)} \iint_{D} \text{div} \mathbf{F} \, dA
\]

\Rightarrow \text{Area}(D') - \text{Area}(D) \approx \Delta t \cdot \iint_{D} \text{div} \mathbf{F} \, dA.

which is also determined by fluid across boundary \(C\) in time \(\Delta t\) \(\approx \Delta t \cdot \oint_{C} (\mathbf{F} \cdot \mathbf{n}) \, ds\)

will be generalized later to \(\mathbb{R}^3\) case, talking about flux across a surface

(or how fast the volume expanding)