Recall \[ \iiint_R f \, dV = \iint_R f \, r \, dr \, d\theta \]

Cylindrical \[ dV = r \, dr \, d\theta \, dz \]

\[ x = r \cos \phi \]
\[ y = r \sin \phi \]
\[ z = z \]

Example 7.8: Consider cylinder \( x^2 + y^2 = 1 \)

below \( z = 4 \)

above \( z = 1 - x^2 - y^2 = 1 - r^2 \)

\[ p(x, y, z) = \sqrt{x^2 + y^2} = r \] density

Find total mass \( \iiint_R p \, dV \).

- All (three) different ways to start (slice).

1. By \( z \) (fix height)
2. By \( r \) (fix radius)
3. By \( \theta \) (fix angle)

We will choose to fix \( r \) first (ii) (check textbook for fixing \( \theta \) first)

\[ \int_0^4 \int_0^{2\pi} \int_0^{1-r^2} r \cdot p \, r \, d\theta \, dz \, dr \]

(Fix \( r \), then \( z \), then \( \theta \))
\[ \int_0^1 \int_{r^2}^{4} \int_0^{2\pi} r^2 \, d\theta \, dz \, dr = \int_0^1 \int_{1-r^2}^{4} 2\pi r^2 \, dz \, dr \]
\[ = \int_0^1 2\pi r^2 \, \left[ z \right]_{r^2}^{4} \, dr = 2\pi \int_0^1 r^2 (4 - (1-r^2)) \, dr \]
\[ = 2\pi \int_0^1 (3r^2 + r^4) \, dr = 2\pi \left( \frac{r^3}{3} + \frac{r^5}{5} \right) \bigg|_0^1 = \frac{12\pi}{5} \]

\textbf{Spherical}

\[ 0 \leq \rho \quad x = \rho \sin \phi \cos \theta \]
\[ 0 \leq \theta < 2\pi \quad y = \rho \sin \phi \sin \theta \]
\[ 0 \leq \phi \leq \pi \quad z = \rho \cos \phi \]

\[ dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]

\textbf{EX} Volume of sphere of radius 1

\[ \iiint_R dV = \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]
\[ = \int_0^{2\pi} \int_0^\pi \frac{1}{3} \rho^3 \sin \phi \, d\theta \, d\phi = \frac{2\pi}{3} \int_0^\pi \sin \phi \, d\phi \]
\[ = \frac{2\pi}{3} \left( -\cos \phi \right) \bigg|_0^\pi = \frac{2\pi}{3} \left( 1 - (0) \right) \]
\[ = \frac{4\pi}{3} \]

\textit{Same answer, much easier to do in spherical coord.}
§ 15.9 Changing Coord.

(Not required for exam)

Goal: Want some transform $T(u,v) \rightarrow (x,y)$ to make integral easier to compute.

$$\iint f \, dx \, dy = \iint f \cdot |A| \, du \, dv.$$ 

Linear transform, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible

$T(u,v) = A \begin{pmatrix} u \\ v \end{pmatrix} = (au + bv, cu + dv)$

In general, if $T(u,v) = (g(u,v), h(u,v))$

then

$$\iint f \, dx \, dy = \iint F \cdot |J| \, du \, dv.$$ 

Linear $\Rightarrow J = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}$