### Multivariable Integration

**Before:** Integrating along curve

\[ \int_c f \, ds, \quad \int_c F \, dr \]

**Now:** Integrating along surface

\[ \int_S f \, dA \]

**(Before)**

\[ f \]

Area = \[ \int_a^b f \, dx \]

**(Now)**

(2-variable)

Volume = \[ \iiint_R f(x, y) \, dA \]

**Volume**

\[ d(Area) \]

w.r.t. Area

**Recall**

Set \( m_i = \min \left\{ f \right\} \) at \( i \)-th interval

\( M_i = \max \left\{ f \right\} \) at \( i \)-th interval
Then \( \sum \min_i \Delta x \leq \int_a^b f(x) \, dx \leq \sum \max_i \Delta x. \)

Take \( \Delta x \to 0 \), if \( f \) continuous, two bounds \( \to \int_a^b f(x) \, dx. \)

Now cut the base into small squares to approximate!

Each square has area \( \Delta x \Delta y \).

Let \( \min_i \) be min. of \( f \) at \( i \)-th box and \( \max_i \) be max of \( f \).

\[
\Rightarrow \sum_{\text{all squares}} \min_i (\Delta x \Delta y) \leq \iint_R f(x,y) \, dA \leq \sum_{\text{all squares}} \max_i (\Delta x \Delta y) .
\]

Approximate volume of these boxes.

- If \( f \) continuous, take \( \Delta x, \Delta y \to 0 \), then again two bounds \( \to \iiint_R f(x,y) \, dA \).
To compute \( \iiint_R f(x,y) \, dA \), reduce to 1-var integrals by slicing!

- Let \( A(y) \) be Area of the cross section with the given \( y \) coord.
- Volume of slice \( \approx A(y) \, dy \)
- Sum up we have

\[
\iiint_R f(x,y) \, dA = \int_c^d A(y) \, dy \\
= \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy \\
= \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx.
\]

(ex) \( f(x,y) = x^2 + y^2 + 5 \) \hspace{1cm} R = \begin{array}{|c|c|}
(a,0) & (a,1) \\
(0,0) & (1,0) \\
\end{array}

\[
\text{Vol} = \iiint_R f(x,y) \, dA = \int_0^1 \left( \int_0^4 (x^2 + y^2 + 5) \, dx \right) \, dy \\
= \int_0^1 \left[ \left( \frac{x^3}{3} + (y^2 + 5)x \right) \right]_{x=0}^{x=1} \, dy \\
= \int_0^1 (\frac{1}{3} + y^2 + 5) \, dy = \left[ \frac{1}{3}y + \frac{1}{3}y^3 \right]_0^1 = \frac{17}{3}
\]
More meanings of double integral.

1. Average of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \)
on (base) \( R \)
   \[
   = \frac{1}{\text{Area}(R)} \iint_R f \, dA
   \]

2. Surface integral of \( f \)
   \( R \) made of material with density \( p : \mathbb{R}^2 \rightarrow \mathbb{R} \).
   \( \Rightarrow \) Mass \( = \iint_R p \, dA \).

3. Set \( f \equiv 1 \)
   \( \Rightarrow \iint_R 1 \, dA = \text{Area of } R \).