3.16.1 Vector Fields

Vector fields are functions \( \vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) (or \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)).

- Each pt in \( \mathbb{R}^2(\mathbb{R}^3) \) is associated with a vector.
- Indicating flow/wind/force field.
- Gravity/electric/magnetic field.

\( \nabla \) gradient is a vector field!

\( f : \mathbb{R}^2 \rightarrow \mathbb{R} \), \( \nabla f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\( f(x, y) = x^2 + 2y^2 \Rightarrow \nabla f = (2x, 4y) \)

\( \vec{F}(x, y) = (1, x) \)

\( \vec{F}(x, y, z) = (0, 0, x) \)
Integrating along curves.

Recall \( C \): curve in \( \mathbb{R}^2 \), \( \vec{r}(t) \), \( f(\vec{r}(t)) \) \( c \)

\[
\text{length} = \int_{a}^{b} |\vec{r}'(t)| \, dt
\]

Now \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) integrate \( f \) along \( C \).

\[
\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, |\vec{r}'(t)| \, dt
\]

Another form: \( \vec{r}'(t) = (x'(t), y'(t)) \)

\[
\int_C f \, ds = \int_a^b f(x(t), y(t)) \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt
\]

Meaning

1. Area

Recall \( \text{cal} \) \( I \), \( f \geq 0 \)

\[
\int_a^b f(x) \, dx \text{ is the area under } f
\]

Now \( f \geq 0 \)

\[
\int_C f \, ds \text{ is the area under } f \text{ along curve } C
\]
2. **Mass**

If \( \rho = \) density of the material that rope made of.

\[ C: \text{curve is a rope} \]

Mass of rope = \( \int_C \rho \, ds \).

3. **Average**

\( f: \mathbb{R} \to \mathbb{R} \) or \( [a, b] \)

Average of \( f \) on \([a, b]\) is

\[ \frac{1}{b-a} \int_a^b f(x) \, dx. \]

Average of \( f \) along \( C \)

is \( \frac{1}{\text{length } C} \int_C f \, ds \).

\[ \text{Find average of } f(x,y) = 2x + y \text{ on } \vec{r}(t) = (\cos t, \sin t); 0 \leq t \leq \frac{\pi}{2} \]

Average = \( \frac{1}{\text{length } C} \int_C f \, ds = \frac{\int_0^{\pi/2} f(\vec{r}(t)) \cdot |\vec{r}'(t)| \, dt}{2\pi/4} \)

\[ = \frac{2}{\pi} \left( \int_0^{\pi/2} (2\cos t + \sin t) \cdot 1 \, dt \right) = \frac{2}{\pi} \left( 2\sin t - \cos t \right) \bigg|_0^{\pi/2} \]

\[ = \frac{2}{\pi} \left[ 2 - 0 - (0-1) \right] = \frac{6}{\pi}. \]
Recall \( \vec{r}(t+\Delta t) \approx \vec{r}(t) + \Delta t \vec{r}'(t) \)  

Linear approximation  

Use segments to approximate curve  

\[
\int_C f ds \approx \sum_{i=0}^{n-1} f(\vec{r}(t_i)) \cdot (\text{length of segment from } t_i \text{ to } t_{i+1}) 
\]

\[
\approx \sum f(\vec{r}(t_i)) \cdot \Delta t \cdot |\vec{r}'(t_i)| 
\]

Take \( \Delta t \to 0 \)  

\[= \int_C f ds. \]