Averaged Template Matching Equations

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Abstract. By exploiting an analogy with averaging procedures in fluid dynamics, we present a set of averaged template matching equations. These equations are analogs of the exact template matching equations that retain all the geometric properties associated with the diffeomorphism group, and which are expected to average out small scale features and so should, as in hydrodynamics, be more computationally efficient for resolving the larger scale features. From a geometric point of view, the new equations may be viewed as coming from a change in norm that is used to measure the distance between images. The results in this paper represent first steps in a longer term program: what is here is only for binary images and an algorithm for numerical computation is not yet operational. Some suggestions for further steps to develop the results given in this paper are suggested.

1 Introduction

1.1 Previous Work

Deformable template matching is a technique for comparing images with applications in computer vision, medical imaging and other fields. It has been reported on extensively in the literature. See for example, Younes (19), Trouvé (17, 18), Grenander and Miller (5) and the references therein.

Template matching is based on the notion of computing a deformation induced distance between two images. The "energy" required to do a deformation that takes one image to the other defines the distance between them. The deformations are often taken to be diffeomorphisms of the image rectangle, i.e smooth maps with smooth inverse. The energy can be defined using various metrics on the space of diffeomorphisms. In addition to diffeomorphisms, which are merely a change of coordinates of the underlying image rectangle, one can also allow changes to the pixel values. Trouvé (17, 18) develops such a theory and gives several numerical examples. He gives conditions on the metric that are sufficient to make the space of deformations a complete metric space. He works with a subgroup of homeomorphisms as the space of deformations and allows pixel value changes by using a semidirect product with a group that acts on the pixel values. The paper by Dupuis, Grenander and Miller (3) also derives conditions for existence of template matching solutions.

Recently a partial differential equation for template matching was derived by Ratnanather, Baigent, Mumford and Miller (15), for both exact and inexact matching. In exact matching the two images being compared have to be

diffeomorphic and in inexact matching they need not be. Their derivation was done using Euler-Poincaré reduction theory and also using classical calculus of variations. For an early version, which does not use the Euler-Poincaré theory, see Mumford (14). For Euler-Poincaré reduction theory, see Marsden and Ratiu (8, Chapters 1 and 13). This technique is useful for computing Euler-Lagrange equations when the Lagrangian is invariant under the action of some Lie group. For example, it is possible to do a variational derivation of the Euler equations of rigid bodies and fluid mechanics using Euler-Poincaré reduction theory.

In their most general form as given in Mumford (14), the exact template matching equations (TME) depend on the choice of a self-adjoint operator that appears in the definition of the metric on the group of diffeomorphisms. When this metric is L^2 we will refer to the equations as L^2-TME.

1.2   Contributions

In this paper we derive the isotropic averaged template matching equations, (H^1_α-ATME), which we hope will be a version of the exact template matching equations that average out small scale features, yet retain the larger scale features. The H^1_α refers to a weighted Sobolev metric that we use instead of the L^2 metric on the group of diffeomorphisms.

Thus the H^1_α-ATME are derived by making a special choice for the self-adjoint operator that appears in the derivation of Ratnanather, Baigent, Mumford and Miller (15). We expect that the averaged equations and even their anisotropic counterparts may also be of interest in computer vision. These might allow template matching while ignoring features smaller than a chosen size (α in equation (13)). The H^1_α-ATME derivation was inspired by recent work on Lagrangian averaged equations in fluid mechanics as described in Marsden and Shkoller (10) and references there in.

By analogy with fluid mechanics the H^1_α-ATME may be much more amenable to numerical solution than the L^2-TME. Finally, by allowing the ignorable feature size to vary it may be possible to perform template matching more robustly using a multiscale approach.

1.3   Overview

We first set up the framework of template matching. For this paper, the main task of template matching reduces to defining distance between binary images. After describing the framework we give the definitions and facts that are needed for our derivation. These preliminaries include a brief summary of Euler-Poincaré reduction in Section 3.3. Before giving a derivation of H^1_α-ATME we repeat the derivation of the TME of Mumford (14) for the special case of the L^2 metric. The main result in this paper is the derivation of the isotropic averaged template matching equations for the exact matching case and this derivation is in Section 6.

2   Our Framework

The basic element of template matching is the computation of a distance between two images and the computation of a deformation that takes one image to the other. For concreteness and simplicity we will limit our attention to binary images, i.e. an image will be a characteristic function on some bounded open subset M of $\mathbb{R}^n$, $n \geq 1$. Thus our space of images is $\mathcal{P} = \{f \mid f : M \to \{0, 1\}\}$. We do the derivations for a general n since those are just as easy as derivations for the case $n = 2$. For $n = 2$, M will typically be a rectangle in the plane.

In order to define a metric on $\mathcal{P}$, given two images f and g one finds the "smallest" map $\varphi : M \to M$ such that $f = g \circ \varphi$. In Section 3.2 we show that this smallest map φ induces a pseudometric on $\mathcal{P}$. This approach does not allow one to modify the range of the images, φ is just a change of coordinates for M. A more general framework that allows one to modify the range of the images (i.e. modify the pixel values) by using semidirect products is described in Trouvé (17).

In order to define the "smallest" map φ mentioned above, one must define a metric on the space to which φ belongs. In addition one has a choice of what space to use as the source of the maps φ. These choices and an analysis of their implications require extensive and subtle analysis that uses many mathematical tools. A nice discussion and use of these subtleties is in Trouvé (17). In this paper, to keep things simple, we will ignore these subtleties. We will take the space to which the maps φ belong to be the space of all diffeomorphisms of M fixing its boundary pointwise and denote this space by $\mathrm{Diff}(M)$.

3   Preliminaries

3.1   Facts about $\mathrm{Diff}(M)$

We will require some basic facts about $\mathrm{Diff}(M)$ and its tangent spaces which we state here, some without proof. We will ignore the difficulties associated with defining a differentiable structure on $\mathrm{Diff}(M)$. For details on this, see Ebin and Marsden (4). See also Marsden and Ratiu (8) for an elementary discussion. The most important fact is that a vector in the tangent space of $\mathrm{Diff}(M)$ is a vector field on M. This is stated more formally in Facts 1 and 2 below.

Fact 1. *The tangent space of* $\mathrm{Diff}(M)$ *at the point* $\varphi \in \mathrm{Diff}(M)$, *is the space of all material (i.e Lagrangian) velocity vector fields* $\boldsymbol{V}_\varphi$ *over* φ *on* M *that vanish on the boundary* ∂M *of* M. *This tangent space is denoted by* $T_\varphi(\mathrm{Diff}(M))$.

Fact 2. *The tangent space of* $\mathrm{Diff}(M)$ *at identity* $e \in \mathrm{Diff}(M)$, *i.e.* $T_e(\mathrm{Diff}(M))$ *is the space* $\mathfrak{X}(M)$ *of all spatial (i.e Eulerian) velocity vector fields on* M *that vanish on the boundary* ∂M *of* M.

We also need some facts about the tangent map (derivative) of the action of $\mathrm{Diff}(M)$ acting on itself. Let $\mathrm{Diff}(M)$ act on itself on the right by function

composition. Thus for $\varphi, \eta \in \mathrm{Diff}(M)$ the right action of η on φ is $\varphi \cdot \eta = R_\eta(\varphi) = \varphi \circ \eta$, where R_η denotes right multiplication of the argument by η. Now we compute the tangent lifted action, i.e TR_η which is the derivative of the right action described above. We show that

Fact 3. *If R_η is the right action defined above, then its derivative is*

$$T_\varphi R_\eta(\boldsymbol{V}_\varphi) = \boldsymbol{V}_\varphi \circ \eta$$

for all $\boldsymbol{V}_\varphi \in T_\varphi(\mathrm{Diff}(M))$. This is called the right action by η on $\boldsymbol{V}_\varphi$.

We recall the proof of this standard fact for the reader's convenience.

Proof. The proof essentially consists of checking definitions. Let $\varphi_t \subset \mathrm{Diff}(M)$ be a smooth curve such that $\varphi_0 = \varphi$ and $d/dt|_{t=0}\varphi_t = \boldsymbol{V}_\varphi$. Thus for every $X \in M$, $\partial/\partial t|_{t=0}\varphi(X, t) = \boldsymbol{V}_\varphi(X)$. Then by definition of derivative $T_\varphi R_\eta(\boldsymbol{V}_\varphi) = d/dt|_{t=0}(\varphi_t \circ \eta)$. Thus, $T_\varphi R_\eta(\boldsymbol{V}_\varphi)(X) = d/dt|_{t=0}\varphi_t(\eta(X)) = \boldsymbol{V}_\varphi(\eta(X))$. $\square$

Definition 1. *An inner product on $\mathrm{Diff}(M)$ is said to be right invariant under action by $\mathrm{Diff}(M)$ if $(\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi) = (T_\varphi R_\eta(\boldsymbol{V}_\varphi), T_\varphi R_\eta(\boldsymbol{U}_\varphi)) = (\boldsymbol{V}_\varphi \circ \eta, \boldsymbol{U}_\varphi \circ \eta)$ for all $\varphi, \eta \in \mathrm{Diff}(M)$ and $\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi \in T_\varphi(\mathrm{Diff}(M))$. Here $(\cdot, \cdot)$ denotes an inner product on $\mathrm{Diff}(M)$.*

3.2   Pseudometric Using Diffeomorphisms

We induce a pseudometric (Abraham, Marsden and Ratiu (1)) on the space of images $\mathcal{P}$ from a metric defined on $\mathrm{Diff}(M)$ as follows.

Definition 2. *The positive valued function $\mathrm{d}_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, is called a **pseudometric induced on $\mathcal{P}$ from** $\mathrm{Diff}(M)$, if for any $f, g \in \mathcal{P}$*

$$\mathrm{d}_\mathcal{P}(f, g) = \inf\{d(e, \varphi) \,|\, \varphi \in \mathrm{Diff}(M) \text{ and } f = g \circ \varphi\} \ ,$$

where e is the identity diffeomorphism, $d(e, \varphi)$ is the geodesic distance between e and φ and inf stands for infimum, or greatest lower bound.

As usual, the geodesic distance on $\mathrm{Diff}(M)$, is defined in terms of the inner product on the tangent spaces of $\mathrm{Diff}(M)$, i.e. the Riemannian metric on $\mathrm{Diff}(M)$. One must prove that $\mathrm{d}_\mathcal{P}$ as defined above is actually a pseudometric, i.e. that it satisfies the symmetry and triangle inequality properties as well as the property that $\mathrm{d}_\mathcal{P}(f, f) = 0$ for all $f \in \mathcal{P}$. This is stated in the following Fact 4. See Miller and Younes (11) for a sketch of a proof or Hirani, Marsden and Arvo (6), which is the technical report version of the present paper, for a more detailed proof.

Fact 4. *If the Riemannian metric on $\mathrm{Diff}(M)$ is right invariant under action by $\mathrm{Diff}(M)$, then the function $\mathrm{d}_\mathcal{P}$ of Definition 2 satisfies the pseudometric axioms, namely that*

1. $\mathrm{d}_\mathcal{P}(f, f) = 0$ for all $f \in \mathcal{P}$;

2. $\mathrm{d}_{\mathcal{P}}(f, g) = \mathrm{d}_{\mathcal{P}}(g, f)$ *for all* $f, g \in \mathcal{P}$ *(symmetry) ; and*

3. $\mathrm{d}_{\mathcal{P}}(f, h) \leq \mathrm{d}_{\mathcal{P}}(f, g) + \mathrm{d}_{\mathcal{P}}(g, h)$ *for all* $f, g, h \in \mathcal{P}$ *(triangle inequality)   .*

Furthermore, if the infimum in Definition 2 is achieved, then $\mathrm{d}_{\mathcal{P}}$ *is a metric on* $\mathcal{P}$*, in which case property 1 becomes*

1. $\mathrm{d}_{\mathcal{P}}(f, g) = 0 \iff f = g$ *(definiteness)   .*

One can also show the same result when right invariance is replaced by left invariance or bi-invariance.

Thus to compute $\mathrm{d}_{\mathcal{P}}(f, g)$ we need to find the smallest diffeomorphism φ in $\mathrm{Diff}(M)$ such that $f = g \circ \varphi$. The definition of *smallest* depends on the chosen metric on $\mathrm{Diff}(M)$. The typical strategy for this is to find a geodesic on $\mathrm{Diff}(M)$, from the identity map to the unknown diffeomorphism φ. The unknown diffeomorphism satisfies the constraint $f = g \circ \varphi$. Any such φ will be the smallest, since it will be the diffeomorphism closest to identity, which also satisfies $f = g \circ \varphi$.

There may or may not be many such smallest diffeomorphisms, but it is sufficient to find one, in order to solve the template matching problem. The lack of uniqueness, if present, may have practical implications for numerical solvers, which is an issue we have not yet addressed. Moreover, there may not exist any such φ ; for example this is the case when the image corresponding to f is not homeomorphic to that corresponding to g. The existence issue also requires further investigation. Trouvé (17, 18) gives conditions that a metric must satisfy for existence and uniqueness of minimizers in inexact matching.

3.3   Euler-Poincaré Reduction

We now recall some facts about Euler-Poincaré reduction theorem that we will need. For details, see Chapter 13 of Marsden and Ratiu (8). Euler-Poincaré reduction is useful in mechanics. For example, it is possible to do a variational derivation of the Euler equations of rigid bodies and fluid mechanics using Euler-Poincaré reduction theory. Consider a Lagrangian L i.e. a map $L : T(\mathrm{Diff}(M)) \to \mathbb{R}$, so L is a function of $\varphi \in \mathrm{Diff}(M)$ and $\dot{\varphi} \in T_{\varphi}(\mathrm{Diff}(M))$. If this Lagrangian is invariant under right action by $\mathrm{Diff}(M)$ then we can use the Euler-Poincaré reduction theorem (Marsden and Ratiu (8) Theorem 13.5.3). According to this theorem, the following two statements are equivalent :

1. The variational principle $\delta \int_a^b L(\varphi(X, t), \dot{\varphi}(X, t)) \, dt = 0$ holds for variations of curves $\varphi(X, t)$ with fixed end points, i.e. for $\delta\varphi(X, a) = \delta\varphi(X, b) = 0 \ \forall X \in M$;

2. The variational principle $\delta \int_a^b l(\boldsymbol{u}(x, t)) \, dt = 0$ holds on $\mathfrak{X}(M)$, i.e. on the tangent space at the identity of $\mathrm{Diff}(M)$, using variations of the form $\delta\boldsymbol{u} = \dot{\boldsymbol{w}} + [\boldsymbol{w}, \boldsymbol{u}]_L$. This is called the reduced variational principle.

Here $\dot{\varphi}(X, t) = \partial/\partial t \varphi(X, t)$ (keeping X fixed). The vector $\boldsymbol{u}$ is the tangent vector $\dot{\varphi}$ moved to identity $e \in \mathrm{Diff}(M)$ by right action by φ_t^{-1}, i.e. $\boldsymbol{u} = \dot{\varphi} \circ$

φ_t^{-1}. Subscript t here denotes fixed time t not derivative w.r.t t. More precisely, if $x = \varphi(X, t) = \varphi_t(X)$ then $\boldsymbol{u}(x, t) = \dot{\varphi}(\varphi_t^{-1}(x), t)$. By this we mean that first the time derivative of $\varphi(X, t)$ is computed keeping X fixed and *then* one substitutes $X = X(x, t) = \varphi_t^{-1}(x)$ into the resulting expression. The function $l : T_e(\text{Diff}(M)) \to \mathbb{R}$ is simply the restriction of L to the tangent space at identity e.

The vector $\boldsymbol{w}$ is the vector $\delta\varphi$ moved to identity in $\text{Diff}(M)$. Thus $\boldsymbol{w} = \delta\varphi_t \circ \varphi_t^{-1}$. The notation $[\boldsymbol{w}, \boldsymbol{u}]_L \equiv (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}$ is the Jacobi Lie bracket. Note that $\boldsymbol{w}(x, a) = \boldsymbol{w}(x, b) = 0 \ \forall x \in M$ since $\delta\varphi(X, a) = \delta\varphi(X, b) = 0 \ \forall X \in M$.

3.4   Gauss-Green Theorem and Its Corollary

We will need some basic facts from vector calculus, for the derivation of the L^2-TME and H_α^1-ATME. We state these facts here.

Fact 5. (Gauss-Green Theorem) *Let M be an open bounded subset of $\mathbb{R}^n$ and suppose that the boundary ∂M is C^1. Suppose $u \in C^1(\overline{M})$. Then*

$$\int_M \frac{\partial u}{\partial x_i} \, dx = \int_{\partial M} u\nu^i \, dS$$

for all $i = 1, \ldots, n$ and where $\hat{\nu} = (\nu^1, \ldots, \nu^n)$ is the outward pointing unit normal field on ∂M.

A simple corollary of the Gauss-Green theorem is the following fact which we will use several times.

Fact 6. *Let M be an open bounded subset of $\mathbb{R}^n$ and suppose that ∂M is C^1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vector fields on $\overline{M}$. Then*

$$\int_M \text{div} \, \boldsymbol{v}\langle \boldsymbol{u}, \boldsymbol{w}\rangle + \langle \boldsymbol{u}, (\boldsymbol{v} \cdot \nabla)\boldsymbol{w}\rangle + \langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}, \boldsymbol{w}\rangle \, dx = \int_{\partial M} \langle \boldsymbol{u}, \boldsymbol{w}\rangle \langle \boldsymbol{v}, \hat{\nu}\rangle \, dS \quad (1)$$

where $\hat{\nu} = (\nu^1, \ldots, \nu^n)$ is the outward pointing unit normal field of ∂M.

Proof. We will use the Gauss-Green theorem (Fact 5) stated above. By this theorem we have that for all i, j in $\{1, \ldots, n\}$, $\int_M \partial/\partial x_j (u^i w^i v^j) dx = \int_{\partial M} u^i w^i v^j \nu^j \, dS$. Thus

$$\sum_{i,j=1}^n \int_M u^i w^i \frac{\partial v^j}{\partial x_j} + u^i v^j \frac{\partial w^i}{\partial x_j} + w^i v^j \frac{\partial u^i}{\partial x_j} dx = \sum_{i,j=1}^n \int_{\partial M} u^i w^i v^j \nu^j dS \ ,$$

which proves equation (1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4   Metrics on Diff(M)

We need to define two different Riemannian metrics on $\text{Diff}(M)$, i.e. inner products on its tangent spaces. One is for the L^2-TME derivation and the other one is for the H_α^1-ATME derivation.

Definition 3. *Let $\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi \in T_\varphi(\mathrm{Diff}(M))$, i.e. $\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi$ are tangent vectors, tangent to $\mathrm{Diff}(M)$ at the point $\varphi \in \mathrm{Diff}(M)$. Let $J(\varphi)(X)$ be the determinant of the derivative of (i.e. the Jacobian determinant of) φ evaluated at point $X \in M$. The L^2 Riemannian metric on $\mathrm{Diff}(M)$ we will use for the L^2-TME derivation is defined as*

$$(\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{L}^2} \equiv \int_M \langle \boldsymbol{V}_\varphi(X), \boldsymbol{U}_\varphi(X) \rangle J(\varphi)(X)\, dX \ . \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

The H^1_α metric on $\mathrm{Diff}(M)$ is defined by first defining it on the tangent space at identity and then extending it to all of $\mathrm{Diff}(M)$ right invariantly.

Definition 4. *Let $\boldsymbol{v}, \boldsymbol{u}$ be vectors in the tangent space at identity $e \in \mathrm{Diff}(M)$ i.e. $\boldsymbol{v}, \boldsymbol{u} \in T_e(\mathrm{Diff}(M))$. For any $\alpha > 0$, $\alpha \in \mathbb{R}$, define*

$$(\boldsymbol{v}, \boldsymbol{u})_{\mathrm{H}^1_\alpha} \equiv \int_M \langle \boldsymbol{v}(x), \boldsymbol{u}(x) \rangle + \alpha^2 \sum_{i=1}^n \langle \mathbf{D}_i\, \boldsymbol{v}(x), \mathbf{D}_i\, \boldsymbol{u}(x) \rangle\, dx \ . \tag{3}$$

where $\mathbf{D}_i\, \boldsymbol{v} = \partial/\partial x_i(\boldsymbol{v}(x))$. The inner products inside the integral are the standard inner products (dot products) on $\mathbb{R}^n$.

To compute the inner product at a point $\varphi \in \mathrm{Diff}(M)$ different from identity, define $(\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{H}^1_\alpha} = (\boldsymbol{V}_\varphi \circ \varphi^{-1}, \boldsymbol{U}_\varphi \circ \varphi^{-1})_{\mathrm{H}^1_\alpha}$. Note that $\boldsymbol{V}_\varphi \circ \varphi^{-1}, \boldsymbol{U}_\varphi \circ \varphi^{-1} \in T_e(\mathrm{Diff}(M))$ because right action by φ^{-1} moves the vectors at φ to identity on $\mathrm{Diff}(M)$. Thus we can use (3) to compute the inner product. Note that we defined the L^2 inner product at a general point of $\mathrm{Diff}(M)$ but defined the H^1_α inner product at identity and showed how it can be computed at a general point. This is done for simplicity. The expression for the L^2 inner product at a general point is simple but not so for the H^1_α inner product. We will write the corresponding norms as follows. The L^2 norm of $\boldsymbol{U}_\varphi \in T_\varphi(\mathrm{Diff}(M))$ is written as $\|\boldsymbol{U}_\varphi\|_{\mathrm{L}^2} \equiv (\boldsymbol{U}_\varphi, \boldsymbol{U}_\varphi)^{1/2}_{\mathrm{L}^2}$ and similarly for the H^1_α norm.

The geodesic distance between φ_a and φ_b on $\mathrm{Diff}(M)$ is defined as

$$d(\varphi_a, \varphi_b) \equiv \inf \left\{ \int_a^b (\dot{\varphi}(t), \dot{\varphi}(t))^{1/2}\, dt \ \middle|\ \varphi(a) = \varphi_a, \varphi(b) = \varphi_b \right\}$$

and the infimum is taken over all smooth parametric curves $\varphi : [a, b] \to \mathrm{Diff}(M)$ from φ_a to φ_b. Here $(\cdot, \cdot)$ is any Riemannian metric on $\mathrm{Diff}(M)$. Thus $d(\varphi_a, \varphi_b) = \int_a^b (\dot{\varphi}(t), \dot{\varphi}(t))^{1/2}\, dt$, where φ is the curve between the endpoints that makes $\int_a^b (\dot{\varphi}(t), \dot{\varphi}(t))^{1/2}\, dt$ stationary.

The same curve makes the functional $\int_a^b (\dot{\varphi}(t), \dot{\varphi}(t))\, dt$ stationary. This fact is sometimes stated as "minimizing length is the same as minimizing kinetic energy". One way to prove this fact is by computing the Euler- Lagrange equations for both the integrals and noting that the equations are the same.

4.1   Right Invariance of L² Metric

We now prove the right invariance property of the L^2 metric in Definition 3. This property is crucial for application of the Euler-Poincaré reduction theorem.

Claim 1. *The metric defined by (2) is right invariant under action of the group* $\mathrm{Diff}(M)$ *acting on* $\mathrm{Diff}(M)$, *i.e. for any* $\eta \in \mathrm{Diff}(M)$

$$(T_\varphi R_\eta(\boldsymbol{V}_\varphi), T_\varphi R_\eta(\boldsymbol{U}_\varphi))_{\mathrm{L}^2} = (\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{L}^2} \ .$$

Proof. By the computation of the tangent lifted group action in Fact 3, this is equivalent to showing that $(\boldsymbol{V}_\varphi \circ \eta, \boldsymbol{U}_\varphi \circ \eta)_{\mathrm{L}^2} = (\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{L}^2}$. The left hand side above is equal to $\int_M \langle \boldsymbol{V}_\varphi(\eta(X)), \boldsymbol{U}_\varphi(\eta(X)) \rangle J(\varphi \circ \eta)(X) \, dX$. Note that the argument $\varphi \circ \eta$ of J is the base point of $T_\varphi R_\eta(\boldsymbol{V}_\varphi)$ and $T_\varphi R_\eta(\boldsymbol{U}_\varphi)$ as required by the definition of the metric. Using chain rule for $J(\varphi \circ \eta)(X)$ the above integral becomes $\int_M \langle \boldsymbol{V}_\varphi(\eta(X)), \boldsymbol{U}_\varphi(\eta(X)) \rangle J(\varphi)(\eta(X)) J(\eta)(X) \, dX$. Now use the change of variable $Y = \eta(X)$. By the change of variables theorem then $dY = J(\eta)(X) dX$ and the above integral becomes $\int_{\eta(M)} \langle \boldsymbol{V}_\varphi(Y), \boldsymbol{U}_\varphi(Y) \rangle J(\varphi)(Y) \, dY$. Since $\eta(M) = M$ the above is equal to $(\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{L}^2}$ as desired.   □

We now give the intuition behind the form of the L^2 inner product. Specifically we address the question of why the Jacobian determinant term appears in the L^2 inner product definition (Definition 2). It appears so that the inner product can be made right invariant. Right invariance of the metric on $\mathrm{Diff}(M)$ implies that the induced function $\mathrm{d}_\mathcal{P}$ is a pseudometric (Fact 4). Thus right invariance (or left- or bi-invariance for that matter) is a convenient assumption. Moreover, the distance between two images should not change if they are both distorted by the same change of variables. This also makes the requirement of invariance attractive.

4.2   Right Invariance of H$^1_\alpha$ Metric

Since the H$^1_\alpha$ metric was defined at identity and extended in a right invariant fashion, the check for right invariance is easy. For completeness we give it below.

Claim 2. *The* H$^1_\alpha$ *metric defined by (3) is right invariant under action of the group* $\mathrm{Diff}(M)$ *acting on* $\mathrm{Diff}(M)$, *i.e. for any* $\eta \in \mathrm{Diff}(M)$

$$(T_\varphi R_\eta(\boldsymbol{V}_\varphi), T_\varphi R_\eta(\boldsymbol{U}_\varphi))_{\mathrm{H}^1_\alpha} = (\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{H}^1_\alpha} \ .$$

Proof. By Fact 3, it is enough to show that $(\boldsymbol{V}_\varphi \circ \eta, \boldsymbol{U}_\varphi \circ \eta)_{\mathrm{H}^1_\alpha} = (\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{H}^1_\alpha}$ for all $\eta \in \mathrm{Diff}(M)$. But by definition of H$^1_\alpha$,

$$\begin{aligned}(\boldsymbol{V}_\varphi \circ \eta, \boldsymbol{U}_\varphi \circ \eta)_{\mathrm{H}^1_\alpha} &= (\boldsymbol{V}_\varphi \circ \eta \circ (\varphi \circ \eta)^{-1}, \boldsymbol{U}_\varphi \circ \eta \circ (\varphi \circ \eta)^{-1})_{\mathrm{H}^1_\alpha} \\ &= (\boldsymbol{V}_\varphi, \boldsymbol{U}_\varphi)_{\mathrm{H}^1_\alpha} \ .\end{aligned}$$

□

5   Template Matching Equations

Both the exact and inexact TME, using Euler-Poincaré reduction theory, and
also using classical calculus of variations, were derived recently, and communi-
cated to us by Ratnanather, Baigent, Mumford and Miller (15). An early version
without the use of Euler-Poincaré theory appears in Mumford (14). We should
note that although Trouvé (17) does not mention Euler-Poincaré reduction, and
does not give a PDE for template matching explicitly, he was certainly aware
of, and used the idea of moving back and forth between the tangent space at
identity and a general point of $\mathrm{Diff}(M)$. For completeness, we now give the
Euler-Poincaré derivation of the L^2-TME in our notation. We do the derivation
of the exact equations, i.e. it is assumed that the two images being compared
are diffeomorphic.

5.1   Derivation of Exact L²-TME

We have seen in Claim 1 that the L^2 metric of Definition 3 is right invariant.
Thus if we define a Lagrangian $L : T(\mathrm{Diff}(M)) \to \mathbb{R}$ as

$$L(\varphi(t), \dot\varphi(t)) = \frac{1}{2}(\dot\varphi(t), \dot\varphi(t))_{\mathrm{L}^2} \tag{4}$$

then it will also be right invariant under that action of $\mathrm{Diff}(M)$.

Let $\varphi : [0,1] \to \mathrm{Diff}(M)$ be a smooth parameterized curve in $\mathrm{Diff}(M)$ be-
tween the points $e = \varphi(0)$ (identity map) and $\varphi(1) \in \mathrm{Diff}(M)$. The point $\varphi(1)$ is
such that $f = g \circ (\varphi(1))$ for the given images f and g. Take a smooth family of
curves φ_ϵ with the same end points $\varphi(0)$ and $\varphi(1)$ and such that $\varphi_0 = \varphi$. Define
the variations of the curve φ to be the vector field $\delta\varphi = d/d\epsilon|_{\epsilon=0}\varphi_\epsilon$ along φ.

Consider the variational principle $\delta \int_0^1 L(\varphi(t), \dot\varphi(t))\, dt = 0$, with the above
variations. By the discussion in Section 3.2 the solution of the variational prin-
ciple above is a geodesic on $\mathrm{Diff}(M)$ under the L^2 metric from the identity map
$e \in \mathrm{Diff}(M)$ to the diffeomorphism $\varphi(1)$ which satisfies the condition of match-
ing, i.e. $f = g \circ (\varphi(1))$. Due to the right invariance of the Lagrangian we can use
the Euler-Poincaré reduction theorem (see Theorem 13.5.3, Page 437 of Marsden
and Ratiu (8) and Section 3.3 of this paper). By applying this theorem, we will
get a variational principle on $\mathfrak{X}(M)$ (called the reduced variational principle)
and hence a differential equation in terms of Eulerian veclocity vector fields on
M. These are the exact L^2-TME derived by Ratnanather, Baigent, Mumford
and Miller (15).

The reduced variational principle uses the reduced Lagrangian l which is
a function on the tangent space at identity of $\mathrm{Diff}(M)$, namely on $\mathfrak{X}(M)$, the
space of all spatial or Eulerian velocity vector fields on M. As noted in Section
3.3 the function l is just the restriction of L to $T_e(\mathrm{Diff}(M))$. Furthermore, L
as defined in equation (4) is right invariant under right action of $\mathrm{Diff}(M)$. As a
result $L(\varphi, \dot\varphi) = L(e, \dot\varphi \circ \varphi^{-1}) = l(\boldsymbol{u})$ where $\boldsymbol{u} = \dot\varphi \circ \varphi^{-1}$. Thus by Definition 3

$$l(\boldsymbol{u}) = \frac{1}{2} \int_M \|\dot\varphi(X,t)\|^2 J(\varphi_t)(X)\, dX \ ,$$

where the overdot is time derivative keeping X fixed and the norm $\|\cdot\|$ is the standard norm in $\mathbb{R}^n$. Let $X = \varphi_t^{-1}(x)$. Then $dX = J(\varphi_t^{-1})(x)\, dx$. Thus,

$$l(\boldsymbol{u}) = \frac{1}{2} \int_M \|\dot{\varphi}(\varphi_t^{-1}(x), t)\|^2 J(\varphi_t)(\varphi_t^{-1}(x)) J(\varphi_t^{-1})(x)\, dx$$

$$= \frac{1}{2} \int_M \|\boldsymbol{u}(x, t)\|^2\, dx = \frac{1}{2} \|\boldsymbol{u}\|_{\mathrm{L}^2}^2 \ .$$

Let us call the functional for the reduced variational principle $\mathcal{E}$, where

$$\mathcal{E}(\boldsymbol{u}) = \int_0^1 l(\boldsymbol{u}(t))\, dt = \frac{1}{2} \int_0^1 \|\boldsymbol{u}(x, t)\|_{\mathrm{L}^2}^2\, dt \ .$$

Then $\delta\mathcal{E} = \int_0^1 \int_M \langle \boldsymbol{u}(x, t), \delta\boldsymbol{u}(x, t) \rangle\, dx\, dt$, where the inner product inside the integral is the usual dot product in $\mathbb{R}^n$. Inserting the definition of $\delta\boldsymbol{u}$ from Section (3.3) in this integral and setting the resulting expression to 0 we get

$$\int_0^1 \int_M \langle \boldsymbol{u}(x, t), \dot{\boldsymbol{w}}(x, t) + [\boldsymbol{w}, \boldsymbol{u}]_L(x, t) \rangle\, dx\, dt = 0.$$

Now substitute the definition of $[\boldsymbol{w}, \boldsymbol{u}]_L$ from Section 3.3, or page 20 of Marsden and Ratiu 8. With this the above equation becomes

$$\int_0^1 \int_M \langle \boldsymbol{u}, \dot{\boldsymbol{w}} \rangle - \langle \boldsymbol{u}, (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} \rangle + \langle \boldsymbol{u}, (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \rangle\, dx\, dt = 0 \ . \tag{5}$$

Using integration by parts on the time variable for the first term, and the fact that $\boldsymbol{w}(x, 0) = \boldsymbol{w}(x, 1) = 0 \ \forall x \in M$ (see Section 3.3) implies that

$$\int_0^1 \int_M \langle \boldsymbol{u}(x, t), \dot{\boldsymbol{w}}(x, t) \rangle\, dx\, dt = - \int_0^1 \int_M \langle \dot{\boldsymbol{u}}(x, t), \boldsymbol{w}(x, t) \rangle\, dx\, dt \ . \tag{6}$$

For the second and third term of equation (5) also, the goal is to rewrite those in the form $\langle \cdot, \boldsymbol{w} \rangle$. To bring the second term into the required form, note that $(\boldsymbol{w} \cdot \nabla)\boldsymbol{u} = \mathbf{D}\,\boldsymbol{u} \cdot \boldsymbol{w}$ where $\mathbf{D}$ denotes the spatial derivative. Thus

$$\langle \boldsymbol{u}, (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} \rangle = \langle \boldsymbol{u}, \mathbf{D}\,\boldsymbol{u} \cdot \boldsymbol{w} \rangle = \langle (\mathbf{D}\,\boldsymbol{u})^T \cdot \boldsymbol{u}, \boldsymbol{w} \rangle \ . \tag{7}$$

For the third term, we use Fact 6. From Fact 2 $\boldsymbol{u}|_{\partial M} = 0$ and so the RHS of equation (1) is 0. Thus

$$\int_M \langle \boldsymbol{u}, (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \rangle\, dx = - \int_M \operatorname{div}\boldsymbol{u} \langle \boldsymbol{w}, \boldsymbol{u} \rangle + \langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{w} \rangle\, dx \ . \tag{8}$$

Using equations (6), (7) and (8) in equation (5) one gets that

$$\int_0^1 \int_M -\langle \dot{\boldsymbol{u}}, \boldsymbol{w} \rangle - \langle (\mathbf{D}\,\boldsymbol{u})^T \cdot \boldsymbol{u}, \boldsymbol{w} \rangle - \langle (\operatorname{div}\boldsymbol{u})\boldsymbol{u}, \boldsymbol{w} \rangle - \langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{w} \rangle\, dx\, dt = 0 \ .$$

Then since $\boldsymbol{w}$ is arbitrary it follows that

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\mathbf{D}\,\boldsymbol{u})^T \cdot \boldsymbol{u} + (\operatorname{div}\boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0 \ . \tag{9}$$

The above equation (9) is the template matching equation, for exact matching, i.e. the L²-TME, as communicated to us by Ratnanather, Baigent, Mumford, Miller (15). We have just repeated the derivation in our notation for complete-ness. Note that for all $(x,t) \in M \times [0,1]$, $(\mathbf{D}\,\boldsymbol{u}(x,t))^T \cdot \boldsymbol{u}(x,t) = \frac{1}{2}\nabla(\|\boldsymbol{u}(x,t)\|^2)$ where now, the norm on the RHS is the standard norm in $\mathbb{R}^n$. With this, the L²-TME, equation (9) can be written in an alternative form as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\operatorname{div}\boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{2}\nabla(\|\boldsymbol{u}\|^2) \tag{10}$$

where $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is the unknown time dependent spatial (Eulerian) velocity vector field on M that vanishes on the boundary ∂M of M.

These equations can be written more concisely, in the Lie derivative form as $\partial\beta/\partial t + \mathcal{L}_{\boldsymbol{u}}\beta = 0$ where β is the *one form density* associated with $\boldsymbol{u}$ and where $\mathcal{L}$ is the Lie derivative. In $\mathbb{R}^n$, $\beta = \sum_{i=1}^{n} u^i dx^i \otimes d^n x$. These are the Euler-Poincaré equations associated with the right invariant L² metric of Definition 3 on the diffeomorphism group. The advantage of this form is that it can accomodate other metrics, simply by changing β. This will become clear when we derive the H¹$_\alpha$-ATME in Section 6.1.

6 Averaged Template Matching Equations

We now derive H¹$_\alpha$-ATME, which are a set of averaged template matching equa-tions. These equations are analogs of the exact template matching equations that retain all the geometric properties associated with the diffeomorphism group and which are expected to average out small scale features and so should, as in hy-drodynamics, be more computationally efficient for resolving the larger scale features. From a geometric point of view, the new equations may be viewed as coming from a change in norm that is used to measure the distance between images.

6.1 Derivation of H¹$_\alpha$-ATME

The steps in deriving the H¹$_\alpha$-ATME are almost identical to those used for L²-TME, except that we use the H¹$_\alpha$ metric (Definition 4) on $\operatorname{Diff}(M)$, instead of the L² metric. Thus we start with a Lagrangian $L : T(\operatorname{Diff}(M)) \to \mathbb{R}$ defined as

$$L(\varphi(t), \dot\varphi(t)) = \frac{1}{2}(\dot\varphi(t), \dot\varphi(t))_{\mathrm{H}^1_\alpha} \ . \tag{11}$$

The initial and final value conditions are the same as in the L²-TME case, i.e. $f = g \circ (\varphi(1))$. By Claim 2 this Lagrangian is invariant under the right action of

$\operatorname{Diff}(M)$. Thus, as in Section 5.1, Euler-Poincaré reduction can be applied but with a different norm. The reduced Lagrangian is $l(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}\|_{\mathrm{H}^1_\alpha}^2$. By Definition 4 of the H^1_α norm, this implies that $l(\boldsymbol{u}) = \frac{1}{2}\int_M \langle\boldsymbol{u},\boldsymbol{u}\rangle + \alpha^2 \sum_{i=1}^n \langle\mathbf{D}_i\,\boldsymbol{u},\mathbf{D}_i\,\boldsymbol{u}\rangle\, dx$. The functional that appears in the reduced variational principle is

$$\mathcal{E}(\boldsymbol{u}) = \int_0^1 l(\boldsymbol{u})\, dt = \frac{1}{2}\int_0^1 \int_M \langle\boldsymbol{u},\boldsymbol{u}\rangle + \alpha^2 \sum_{i=1}^n \langle\mathbf{D}_i\,\boldsymbol{u},\mathbf{D}_i\,\boldsymbol{u}\rangle\, dx\, dt\ .$$

Let $\boldsymbol{u}_\epsilon$ be a one parameter family of spatial vector fields on M, depending smoothly on ϵ such that $\boldsymbol{u}_0 = \boldsymbol{u}$ and as usual $\delta\boldsymbol{u} \equiv \partial\boldsymbol{u}_\epsilon/\partial\epsilon|_{\epsilon=0}$. The variations $\delta\mathcal{E}(\boldsymbol{u})$ are given by

$$\delta\mathcal{E}(\boldsymbol{u}) = \int_0^1 \langle\frac{\delta l}{\delta\boldsymbol{u}},\delta\boldsymbol{u}\rangle\, dt\ . \tag{12}$$

We now compute the above expression. The integrand is

$$\langle\frac{\delta l}{\delta\boldsymbol{u}},\delta\boldsymbol{u}\rangle \equiv \frac{d}{d\epsilon}\Big|_{\epsilon=0} l(\boldsymbol{u}_\epsilon) = \int_M \langle\boldsymbol{u}_\epsilon,\frac{\partial\boldsymbol{u}_\epsilon}{\partial\epsilon}\rangle + \alpha^2 \sum_{i=1}^n \langle\frac{\partial\boldsymbol{u}_\epsilon}{\partial x_i},\frac{\partial^2\boldsymbol{u}_\epsilon}{\partial\epsilon\partial x_i}\rangle\Big|_{\epsilon=0}\, dx\ .$$

But

$$\sum_{i=1}^n \langle\frac{\partial\boldsymbol{u}_\epsilon}{\partial x_i},\frac{\partial^2\boldsymbol{u}_\epsilon}{\partial\epsilon\partial x_i}\rangle = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial u_\epsilon^j}{\partial x_i}\frac{\partial^2 u_\epsilon^j}{\partial\epsilon\partial x_i}$$

$$= \sum_{j=1}^n\sum_{i=1}^n \frac{\partial u_\epsilon^j}{\partial x_i}\frac{\partial^2 u_\epsilon^j}{\partial\epsilon\partial x_i} = \sum_{j=1}^n \langle\frac{\partial u_\epsilon^j}{\partial x},\frac{\partial^2 u_\epsilon^j}{\partial x\partial\epsilon}\rangle\ .$$

Now

$$\int_M \sum_{i=1}^n \langle\frac{\partial u_\epsilon^i}{\partial x},\frac{\partial^2 u_\epsilon^i}{\partial x\partial\epsilon}\rangle\Big|_{\epsilon=0}\, dx = \int_M \sum_{i=1}^n \langle\frac{\partial u^i}{\partial x},\frac{\partial(\delta u^i)}{\partial x}\rangle\, dx\ .$$

Then using integration by parts,

$$\int_M \langle\frac{\partial u^i}{\partial x},\frac{\partial(\delta u^i)}{\partial x}\rangle\, dx \equiv \int_M \sum_{j=1}^n \frac{\partial u^i}{\partial x_j}\frac{\partial(\delta u^i)}{\partial x_j}\, dx$$

$$= -\int_M \sum_{j=1}^n \frac{\partial^2 u^i}{\partial x_j^2}\delta u^i\, dx + \int_{\partial M} \sum_{j=1}^n \frac{\partial u^i}{\partial x_j}\delta u^i \nu^j\, dS$$

where $\nu = (\nu^1,\dots,\nu^n)$ is the outward pointing normal field on the boundary ∂M of M. But by Fact 2, $\boldsymbol{u}|_{\partial M} = 0$ so we get that

$$\int_M \langle\frac{\partial u^i}{\partial x},\frac{\partial(\delta u^i)}{\partial x}\rangle\, dx = -\int_M \sum_{j=1}^n \frac{\partial^2 u^i}{\partial x_j^2}\delta u^i\, dx = -\int_M (\Delta u^i)\delta u^i\, dx\ .$$

Thus

$$\langle \frac{\delta l}{\delta \boldsymbol{u}}, \delta \boldsymbol{u} \rangle = \int_M \langle \boldsymbol{u}, \delta \boldsymbol{u} \rangle - \alpha^2 \sum_{i=1}^n (\Delta u^i) \delta u^i \, dx \quad .$$

Substituting this into equation (12) we get that

$$\delta \mathcal{E}(\boldsymbol{u}) = \int_0^1 \int_M \langle \boldsymbol{u}, \delta \boldsymbol{u} \rangle - \alpha^2 \sum_{i=1}^n (\Delta u^i) \delta u^i \, dx \, dt$$

$$= \int_0^1 \int_M \langle \boldsymbol{u} - \alpha^2 \Delta \boldsymbol{u}, \delta \boldsymbol{u} \rangle \, dx \, dt \quad .$$

Now substituting the expression for $\delta \boldsymbol{u}$ from Section 3.3 we get

$$\delta \mathcal{E}(\boldsymbol{u}) = \int_0^1 \int_M \langle \boldsymbol{u} - \alpha^2 \Delta \boldsymbol{u}, \dot{\boldsymbol{w}} - (\boldsymbol{w} \cdot \nabla) \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \rangle \, dx \, dt \quad .$$

We move the derivative operators away from $\boldsymbol{w}$, as in the L^2-TME derivation in Section 5.1, to get an integrand of the form $\langle \cdot, \boldsymbol{w} \rangle$. Because of the arbitrariness of $\boldsymbol{w}$, we get the isotropic averaged H$_\alpha^1$ template matching equations, i.e. the H$_\alpha^1$-ATME as

$$\frac{\partial}{\partial t} \boldsymbol{u} - \alpha^2 \frac{\partial}{\partial t} \Delta \boldsymbol{u} + \boldsymbol{u}(\operatorname{div} \boldsymbol{u}) - \alpha^2 (\operatorname{div} \boldsymbol{u}) \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} -$$
$$\alpha^2 (\boldsymbol{u} \cdot \nabla) \Delta \boldsymbol{u} + (\mathbf{D}\,\boldsymbol{u})^T \cdot \boldsymbol{u} - \alpha^2 (\mathbf{D}\,\boldsymbol{u})^T \cdot \Delta \boldsymbol{u} = 0 \quad , \quad (13)$$

where Δ is the componentwise Laplacian and $\mathbf{D}\,\boldsymbol{u}$ is the spatial derivative of $\boldsymbol{u}$. Here $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is the unknown time dependent spatial (Eulerian) velocity vector field on M which vanishes on the boundary ∂M of M. To make it easier to see the relationship between the structure of the H$_\alpha^1$-ATME and L^2-TME we define $\boldsymbol{v} \equiv (1 - \alpha^2 \Delta) \boldsymbol{u}$. Then the H$_\alpha^1$-ATME equation (13) becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\mathbf{D}\,\boldsymbol{u})^T \cdot \boldsymbol{v} + (\operatorname{div} \boldsymbol{u}) \boldsymbol{v} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} = 0 \quad . \tag{14}$$

The Lie derivative form of these equations is $\partial \beta / \partial t + \mathcal{L}_{\boldsymbol{u}} \beta = 0$ where now β is the one form density associated with $\boldsymbol{v} = (1 - \alpha^2 \Delta) \boldsymbol{u}$.

7   Connections with Fluid Mechanics

We now give the analogy and connections with fluid mechanics which have inspired our present work. We start with the connection between TME and fluid mechanics. The L^2-TME in n spatial dimensions, namely equation (9) or equivalently (10), is a higher dimensional analogue of the inviscid Burger's equation. By this we mean that in one spatial dimension the L^2-TME reduce to the equation $u_t + 3uu_x = 0$ where the subscripts indicate derivatives and $u = u(x,t)$ is

the velocity of the fluid at location x at time t. This fact is mentioned in Mumford (13). Burger's equation, at least in the initial value formulation, can develop shocks. The framework for L^2-TME sets this equation in higher dimensions, and as an initial-final value problem. It is possible that shocks may develop in this formulation also. The well-posedness and possibility of shocks in L^2-TME remain to be checked. However, as in the case of Burger's equation and incompressible fluid mechanics, they are well-posed for short time evolution; see Marsden, Ratiu and Shkoller (9) and Marsden and Shkoller (10) and references therein.

Our H^1_α-ATME, on the other hand, contain diffusive terms which may ameliorate some of the analytical problems of L^2-TME, and may be a reason to expect more stable numerics. In one spatial dimension our H^1_α-ATME, equations (13) reduce to the shallow water equations $u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx}$ or equivalently $v_t + uv_x + 2vu_x = 0$ where $v = u - u_{xx}$ (here α of H^1_α-ATME has been set to unity). These equations are completely integrable and have peaked solitons, as was shown by Camassa and Holm (2). The shallow water equations, like Burger's equation, also have a smooth spray, as was shown by Shkoller (16) and hence one has local existence and uniqueness of geodesics. The L^2-TME are equations of geodesics on $\mathrm{Diff}(M)$ under the right invariant L^2 metric whereas the H^1_α-ATME are equations of geodesics under a right invariant H^1_α metric.

Recently, averaged Euler and Navier-Stokes equations for fluid mechanics have been developed. See for example Holm, Marsden and Ratiu (7) and Marsden, Ratiu and Shkoller (9). Besides having nice analytical properties, preliminary numerical experiments (see Mohseni et al (12) and references therein) of these averaged Euler and Navier-Stokes equations show the possible advantages of an averaging approach in the context of numerical solution of nonlinear equations of fluid mechanics.

There is however a very important difference between the template matching framework and the usual fluid mechanics framework. In template matching the equations have to be solved as an initial-final value problem. Thus one is given image f and it has to be deformed to image g moving the pixels in such a way that the motion of the image during the deformation satisfies the template matching equations. In fluid mechanics one is typically interested in giving some initial velocity and studying how the particles move.

8   Discussion and Future Work

We have named the H^1_α equations *averaged* equations. This is with the expectation that they will have averaging properties like the averaged equations of hydrodynamics described in Marsden and Shkoller (10). But we have not yet done an averaging derivation in the template matching context to see if the solution does indeed allow one to compare images while ignoring features smaller than α. This is the most important task that remains to be done. Furthermore, a condition like $f = g \circ (\varphi(1))$ will have to be modified in the averaging case possibly by preprocessing f and g.

It is our expectation that there will be considerable improvement in the numerical performance of the H_α^1-ATME. This expectation is based on the analogous situation in fluid dynamics in which averaging improves numerical simulation as demonstrated in Mohseni et. al (12) and references therein. We also plan to investigate generalizations of the procedures here using inexact matching and semidirect product theory (used for *inhomogeneous* fluid problems, for example), to generalize beyond binary images. We also want to explore the role of the rotation group and reduction constructions. This is based on the fact that the notion of *shape space* is a technique common to both mechanics and computer vision.

Due to an initial-final value formulation, it appears that an optimization approach is one way to solve template matching type equations, as is done in Trouvé (17, 18). Thus when it comes to computations, the PDEs of template matching (9) or (13) may, or may not be used directly for some applications. In the optimization approach, one goes back to the Lagrangian (material) formulation on the group and finds geodesics directly, by a gradient descent type algorithm for example.

One benefit of the PDE formulation is theoretical, now one knows what equation is being solved. But more importantly, we emphasize that the very reason that we were able to arrive at an averaged version of L^2-TME was because we knew how averaging worked in hydrodynamics by changing the metric on the group $\mathrm{Diff}(M)$, and we knew how to go back and forth between an arbitrary point on the group and identity using Euler-Poincaré theory. Finally, there may be other problems within computer vision or outside it, in which the PDEs are used directly in the computations.

As for the existence of minimizer of the energy functional, Trouvé (17, 18) uses a different definition of distance on $\mathrm{Diff}(M)$. He gives a sufficient condition on the metric on $\mathrm{Diff}(M)$ for the variational problem of template matching to have a minimizer. Roughly speaking, this condition (which is part of what he calls the admissibility criteria) is that the metric must be as strong as the C^1 metric. By Sobolev embedding theorem this implies that an H^s metric will satisfy this condition iff $s > n/2 + 1$. Thus for $n = 2$ or 3, an H^1 or H^2 metric, including the H_α^1 metric will not satisfy this condition. However, note that Trouvé's admissibility criteria is a sufficient condition. In practice in template matching sometimes the metric $\int_M \langle \boldsymbol{u}, \boldsymbol{u} \rangle + (\Delta \boldsymbol{u})^2 dx$ is used as mentioned in Grenander and Miller (5). This metric also does not satisfy the admissibility criteria. This suggests that more work on the theory of existence of minimizer for template matching functionals is needed.

Finally we note that we have only shown that $\mathrm{d}_\mathcal{P}$ is a pseudometric. If the infimum in Definition 2 in not achieved $\mathrm{d}_\mathcal{P}$ is only a pseudometric and there can exist f and g, $f \neq g$, but $\mathrm{d}_\mathcal{P}(f, g) = 0$. What do specific examples (if any) of such f, g pairs look like ? Even more interestingly one can ask if for some metrics on $\mathrm{Diff}(M)$, for example for the H_α^1 metric of Definition 4, $\mathrm{d}_\mathcal{P}$ is a metric. We plan to investigate these questions.

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