Random Triangles
IGL Project Report, Fall 2013

Zehan Chao, Liran Chen, Hao Gao, Mianfeng Liu, Gilad Margalit
A.J. Hildebrand (Faculty Mentor)
University of Illinois at Urbana-Champaign

1 Background

Our work was motivated by the following classical problems:

Problem 1: Broken Stick Problem (1854). *A stick is broken up at two points, chosen at random along its length. What is the probability that the pieces obtained form a triangle?*

Problem 2: Lewis Carroll's Pillow Problem (1893) [?]. *Three points are taken at random on an infinite plane; find the chance of their being the vertices of an* ***obtuse-angled*** *triangle ?*

Problem 3: Sylvester's Four Points Problem (1864) [?]. *What is the probability that four points chosen at random in a given finite region R can form a* ***convex*** *quadrilateral?*

The last problems is equivalent to the following problem on areas of random triangles; see [?].

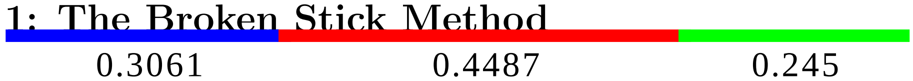
Problem 3': Equivalent form of Sylvester's Four Points Problem (1864). *What is the* ***average area*** *of a random triangle obtained by choosing three vertices at random in a given finite region R?*

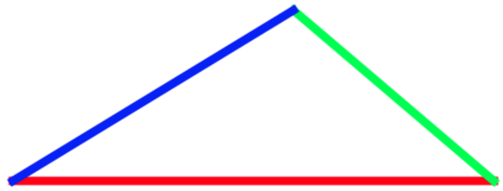
The relation between these two problems is given by the formula

$$P(\text{random quadrilateral in } R \text{ is convex}) = 1 - 4\,\frac{\text{Average area of random triangle in } R}{\text{Area of } R};$$

2 Random Triangle Constructions

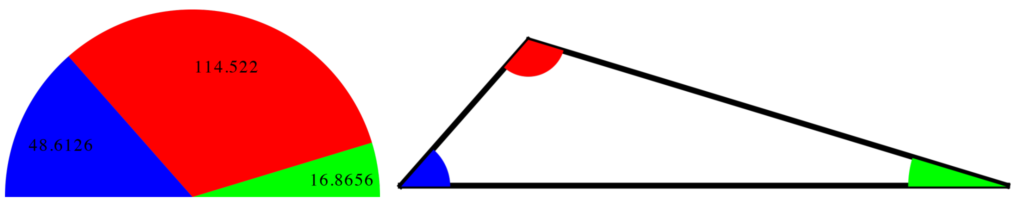
Method 1: The Broken Stick Method





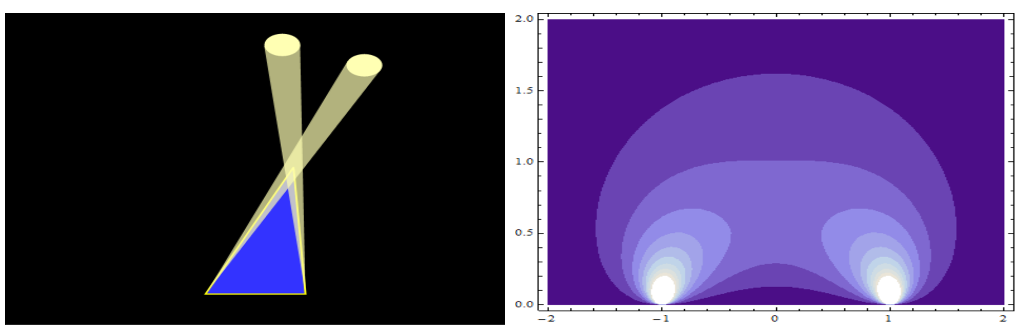
- Take a stick of length 1.
- Pick 2 random points on the stick.
- Break the stick into 3 pieces at these points.
- Form a triangle, if possible.

Method 2: The Broken Angle Method



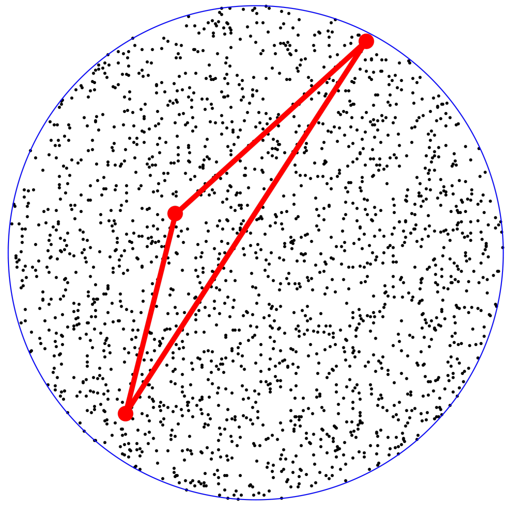
- Break up the angle 180 degrees into 3 angles randomly.
- Form a triangle with perimeter 1 and these 3 angles.

Method 3: The Random Angle Method



- Take a stick of length 1.
- Pick 2 random angles between 0 and 180 degrees.
- Form a triangle with the stick and these angles, if possible.

Method 4: The Random Vertices Method



- Pick 3 points randomly from a unit disk.
- From a triangle with these vertices.

3 Basic Questions

For each of the above constructions of a "random triangle", one can ask the following questions, which are motivated by Problems 1–3 above.

- What is the probability that the pieces form **a triangle**?
- What is the probability that the pieces form an **acute** triangle?
- What is the **average area** of the triangle, if one exists?

The answers to these questions depend on the particular construction method and are summarized in the table below.

Method	Broken Stick	Broken Angle	Random Angle	Random Vertices
Prob. for triangle	**0.25**	**1**	**0.5**	**1**
Prob. for acute triangle*	**0.32**	**0.25**	**0.25**	0.28
Average Area	**0.030**	0.025	∞	0.232
Average Normalized Area	**0.030**	0.025	0.025	0.027
Median Largest Angle	102°	107°	107°	**109°**

- Table shows acute triangle probabilities and average areas for various methods of constructing random triangles.
- *Probabilities are conditional probabilities given a triangle exists.
- Value in **boldface** denotes maximal value in each row.
- Values in **red** were obtained from theoretical formulas. Other values were obtained by Monte Carlo Simulations.

In particular, the table shows that, the Broken Stick Method is *most likely* to produce an acute triangle, while the Random Angle Method triangle generates a triangle with the *largest area*?

4 Generalizations

Results: The Broken Stick Method

- **Probability that the largest angle is $\leq \alpha$:**

$$\begin{cases} \frac{(3-3\cos\alpha)\log(2-2\cos\alpha)-(2\cos\alpha-1)(\cos\alpha-2)}{(1+\cos\alpha)^2} & \text{if } t < \pi/2 \\ \frac{\cos\alpha-2}{\cos\alpha+1} + \frac{(3\cos\alpha-3)\log(\frac{1-\cos\alpha}{2})}{(1+\cos\alpha)^2} & \text{if } t \geq \pi/2 \end{cases}$$

- **Probability that the area is $\leq x$:**

$$2\left(\frac{1}{8} - \int_{\frac{1}{2}}^{1} \sqrt{\max\left(0, \frac{2y^3-5y^2+4y-1-16x^2}{2y-1}\right)}\, dy\right)$$

Results: The Random Angle Method

- **Probability that the largest angle is $\leq \theta$:**

$$\begin{cases} \frac{(3\theta-\pi)^2}{\pi^2} & \text{if } \frac{\pi}{3} < \theta < \frac{\pi}{2} \\ 1 - \frac{3(\theta-\pi)^2}{\pi^2} & \text{if } \frac{\pi}{2} < \theta < \pi \end{cases}$$

- **Probability distribution of the intersection point on the xy-plane:**

$$\frac{4y}{\pi^2(((x-1)^2+y^2)((1+x)^2+y^2))}$$

- **Probability that the area is $\leq t$:**

$$\frac{2\tan^{-1}(t)}{\pi}$$

5 Future Goals

- Solve the remaining theoretical questions in Random Triangle Problem.
- Find generalizations: for example, to higher dimensions or tetrahedra.
- **Demos/animations.** Continue to develop animations and interactive demos for use at Open Houses and high school visits.

6 Additional Activities

- **Homecoming Open House, October 26.** Participated with exhibits and interactive demos.
- **Fall 2013 IGL Open House, December 12.** Participated with exhibits and interactive demos.