1 Background

Our work was motivated by the following classical problems:

Problem 1: Broken Stick Problem (1854). A stick is broken up at two points, chosen at random along its length. What is the probability that the pieces obtained form a triangle?

Problem 2: Lewis Carroll’s Pillow Problem (1893). Three points are taken at random on an infinite plane; find the chance of their being the vertices of an obtuse-angled triangle?

Problem 3: Sylvester’s Four Points Problem (1864). What is the probability that four points chosen at random in a given finite region \( R \) can form a convex quadrilateral?

The last problem is equivalent to the following problem on areas of random triangles; see [\(?\)].

Problem 3’: Equivalent form of Sylvester’s Four Points Problem (1864). What is the average area of a random triangle obtained by choosing three vertices at random in a given finite region \( R \)?

The relation between these two problems is given by the formula

\[
P(\text{random quadrilateral in } R \text{ is convex}) = 1 - 4 \frac{\text{Average area of random triangle in } R}{\text{Area of } R};
\]

2 Random Triangle Constructions

Method 1: The Broken Stick Method

\[
\begin{array}{ccc}
0.3061 & 0.4487 & 0.245 \\
\end{array}
\]
• Take a stick of length 1.
• Pick 2 random points on the stick.
• Break the stick into 3 pieces at these points.
• Form a triangle, if possible.

Method 2: The Broken Angle Method

• Break up the angle 180 degrees into 3 angles randomly.
• Form a triangle with perimeter 1 and these 3 angles.

Method 3: The Random Angle Method

• Take a stick of length 1.
• Pick 2 random angles between 0 and 180 degrees.
• Form a triangle with the stick and these angles, if possible.
Method 4: The Random Vertices Method

- Pick 3 points randomly from a unit disk.
- From a triangle with these vertices.

3 Basic Questions

For each of the above constructions of a “random triangle”, one can ask the following questions, which are motivated by Problems 1–3 above.

- What is the probability that the pieces form a triangle?
- What is the probability that the pieces form an acute triangle?
- What is the average area of the triangle, if one exists?

The answers to these questions depend on the particular construction method and are summarized in the table below.

<table>
<thead>
<tr>
<th>Method</th>
<th>Broken Stick</th>
<th>Broken Angle</th>
<th>Random Angle</th>
<th>Random Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob. for triangle</td>
<td>0.25</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>Prob. for acute triangle*</td>
<td>0.32</td>
<td>0.25</td>
<td>0.25</td>
<td>0.28</td>
</tr>
<tr>
<td>Average Area</td>
<td>0.030</td>
<td>0.025</td>
<td>∞</td>
<td>0.232</td>
</tr>
<tr>
<td>Average Normalized Area</td>
<td>0.030</td>
<td>0.025</td>
<td>0.025</td>
<td>0.027</td>
</tr>
<tr>
<td>Median Largest Angle</td>
<td>102°</td>
<td>107°</td>
<td>107°</td>
<td>109°</td>
</tr>
</tbody>
</table>
Table shows acute triangle probabilities and average areas for various methods of constructing random triangles.

*Probabilities are conditional probabilities given a triangle exists.

Value in **boldface** denotes maximal value in each row.

Values in **red** were obtained from theoretical formulas. Other values were obtained by Monte Carlo Simulations.

In particular, the table shows that, the Broken Stick Method is *most likely* to produce an acute triangle, while the Random Angle Method triangle generates a triangle with the *largest area*?

### 4 Generalizations

#### Results: The Broken Stick Method

- Probability that the largest angle is \( \leq \alpha \):
  \[
  \begin{cases} 
  \frac{(3-3\cos \alpha) \log (2-2\cos \alpha) - (2\cos \alpha-1)(\cos \alpha-2)}{(1+\cos \alpha)^2} & \text{if } t < \pi/2 \\
  \frac{\cos \alpha-2}{\cos \alpha+1} + \frac{(3\cos \alpha-3) \log (1-\cos \alpha)}{(1+\cos \alpha)^2} & \text{if } t \geq \pi/2
  \end{cases}
  \]

- Probability that the area is \( \leq x \):
  \[
  2 \left( \frac{1}{8} - \int_{\frac{t}{2}}^{1} \sqrt{\max \left( 0, \frac{2y^3 - 5y^2 + 4y - 1 - 16x^2}{2y - 1} \right)} \, dy \right)
  \]

#### Results: The Random Angle Method

- Probability that the largest angle is \( \leq \theta \):
  \[
  \begin{cases} 
  \frac{(3\theta - \pi)^2}{\pi^2} & \text{if } \frac{\pi}{3} < \theta < \frac{\pi}{2} \\
  1 - \frac{3(\theta-\pi)^2}{\pi^2} & \text{if } \frac{\pi}{2} < \theta < \pi
  \end{cases}
  \]

- Probability distribution of the intersection point on the xy-plane:
  \[
  \frac{4y}{\pi^2((x-1)^2+y^2)((1+x)^2+y^2)}
  \]

- Probability that the area is \( \leq t \):
  \[
  \frac{2\tan^{-1}(t)}{\pi}
  \]
5 Future Goals

- Solve the remaining theoretical questions in Random Triangle Problem.
- Find generalizations: for example, to higher dimensions or tetrahedra.
- Demos/animations. Continue to develop animations and interactive demos for use at Open Houses and high school visits.

6 Additional Activities

- Homecoming Open House, October 26. Participated with exhibits and interactive demos.

- Fall 2013 IGL Open House, December 12. Participated with exhibits and interactive demos.