Summing Divergent Series: From Euler and Abel to Dirichlet and Beyond

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Divergent Series

Notable enough, however, are the controversies over the series
1 − 1 + 1 − 1 + 1 − 1 + ··· whose sum was given by Leibniz as 1/2, although others disagree.

—Leonhard Euler (1760)

... before Cauchy mathematicians asked not ‘How shall we define
1 − 1 + 1 − 1 + ···’ but ‘What is
1 − 1 + 1 − 1 + ···’ and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.

—G. H. Hardy (1949)

Divergent Series: Formal Definition

The series \( \sum_{n=1}^{\infty} a(n) \) is divergent if \( \sum_{n=1}^{N} a(n) \) does not approach a limit as \( N \to \infty \).

Divergent Series: Summation Methods

The series \( \sum_{n=1}^{\infty} a(n) \) is:
- Abel summable with sum \( S \) if \( \sum_{n=1}^{\infty} a(n)r^n \to S \) as \( r \to 1^- \).
- Dirichlet summable with sum \( S \) if \( \sum_{n=1}^{\infty} \frac{a(n)}{n} \to S \) as \( s \to 0^+ \).
- Lambert summable with sum \( S \) if \( (1 - s) \sum_{n=1}^{\infty} \frac{a(n)r^n}{1-r} \to S \) as \( r \to 1^- \).

Example: Grandi’s Series 1 − 1 + 1 − 1 + ···
- Divergent.
- Abel summable, with sum 1/2.
- Dirichlet summable, with sum 1/2.
- Lambert summable, with sum 1/2.

The Moebius Function

The Moebius function \( \mu(n) \) is defined as:
- \( \mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors} \\ 0 & \text{if } n \text{ has any repeated prime factors} \end{cases} \)

The Moebius function is one of the most important functions in number theory.

The behavior of the Moebius function is closely related to
the Prime Number Theorem and the Riemann Hypothesis.

Lambert Summability

The series \( \sum_{n=1}^{\infty} a(n) \) is Lambert summable with sum \( S \) if \( (1 - s) \sum_{n=1}^{\infty} \frac{a(n)r^n}{1-r} \to S \) as \( r \to 1^- \).

Computations performed on UIUC Campus Computing Cluster using a custom C program.

The Myth: Theoretical vs. Numerical Behavior

The data suggests that \( f(r) \) and \( g(r) \) both have limit \( -2 \) as \( r \to 1^- \), thus \( \sum_{n=1}^{\infty} \mu(n) \) appears to be Abel summable and Lambert summable with sum \(-2\). But this contradicts the known theoretical results (see above).

The Explanation

Analytic Expansion of Moebius Power Series

\[
f(r) \approx 1 - 2 + \frac{12(1-r)}{20} + \frac{120(1-r)^2}{45} + \cdots
\]

Analytic expansion of \( f(r) \) reveals:
- Asymptotically constant term due to pole of Gamma function (observed behavior).
- Unbounded oscillating term with very small coefficient due to zeros of Riemann Zeta function (long range behavior).

References

