Summing Divergent Series: From Euler and Abel to Dirichlet and Beyond

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Divergent Series

Notable enough, however, are the controversies over the series $1-1+1-1+1-\cdots$ whose sum was given by Leibniz as $1/2$, although others disagree.
–Leonhard Euler (1760)

... before Cauchy mathematicians asked not 'How shall we define $1-1+1\cdots$?' but 'What is $1-1+1\cdots$?' and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.
–G. H. Hardy (1949)

GHHardy.j

Divergent Series: Formal Definition

- The series $\sum_{n=1}^{\infty} a(n)$ is **divergent** if

$$\sum_{n=1}^{N} a(n)$$

does **not** approach a limit as $N \longrightarrow \infty$.

Divergent Series: Summation Methods

The series $\sum_{n=1}^{\infty} a(n)$ is

- **Abel summable** with sum S if

$$\sum_{n=1}^{\infty} a(n)r^n \longrightarrow S \text{ as } r \longrightarrow 1-.$$

- **Dirichlet summable** with sum S if

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \longrightarrow S \text{ as } s \longrightarrow 0.$$

- **Lambert summable** with sum S if

$$(1-r)\sum_{n=1}^{\infty} \frac{na(n)r^n}{1-r^n} \longrightarrow S \text{ as } r \longrightarrow 1-.$$

Example: Grandi's Series $1-1+1-1+\cdots$

Grandi's Series is

- *Divergent.*
- *Abel summable, with sum $1/2$.*
- *Dirichlet summable, with sum $1/2$.*
- *Lambert summable, with sum $1/2$.*

Summing the Moebius Series

Moebius Function

- **The Moebius function:** The Moebius function $\mu(n)$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors} \\ 0 & \text{if } n \text{ has any repeated prime factors.} \end{cases}$$

- The Moebius function is one of the most important functions in number theory.
- The behavior of the Moebius function is closely related to the Prime Number Theorem and the Riemann Hypothesis.

Summability of the Moebius series

***Question:** Is the Moebius series*

$$\sum_{n=1}^{\infty} \mu(n) = 1-1-1-1+1+\cdots$$

summable by appropriate summation method, analogous to Grandi's Series?

***Theoretical results:** The Moebius series is*

- *Dirichlet summable, with sum -2.*
- ***Not** Abel summable (Delange, 2000).*
- ***Not** Lambert summable (Delange, 2000).*

Summing the Moebius Series: Numerical Data

Abel Summability

$$f(r) = \sum_{n=1}^{\infty} \mu(n)r^n$$

r	$f(r)$	Source
0.9	-1.1384289	Froberg (1966)
0.99	-1.8867855	Froberg (1966)
0.999	-1.9881049	Froberg (1966)
0.9999	-1.9988015	Froberg (1966)
0.99999	-1.9998804	Froberg (1966)
0.999999	-1.9999878	New
0.9999999	-1.9999945	New

Lambert Summability

$$g(r) = (1-r)\sum_{n=1}^{\infty} \frac{n\mu(n)r^n}{1-r^n}$$

r	$g(r)$	Source
0.9	-1.4015041790	New
0.99	-1.9316138890	New
0.999	-1.9930227310	New
0.9999	-1.9993008648	New
0.99999	-1.999280755	New
0.999999	-1.9999820280	New
0.9999999	-1.9999890043	New

Computations performed on UIUC Campus Computing Cluster using a custom C program.

The Mystery: Theoretical vs. Numerical Behavior

The data suggests that $f(r)$ and $g(r)$ both have limit -2 as $r \longrightarrow 1-$, thus $\sum_{n=1}^{\infty} \mu(n)$ appears to be Abel summable and Lambert summable with sum -2. But this contradicts the known theoretical results (see above).

The Explanation

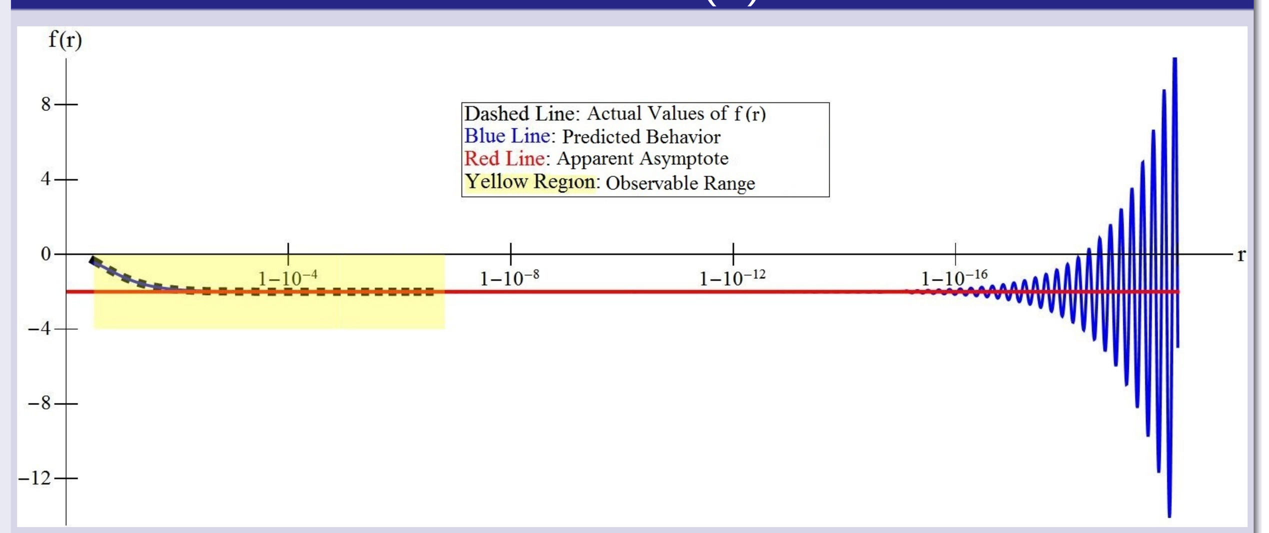
Analytic Expansion of Moebius Power Series

$$f(r) \approx \underbrace{-2+12(1-r)}_{\text{Observed Behavior}} + \underbrace{\frac{0.000000001439}{\sqrt{1-r}}\cos(\cdots)+\cdots}_{\text{Long Range Behavior}}$$

Analytic expansion of $f(r)$ reveals:

- Asymptotically constant term due to pole of Gamma function (observed behavior).
- Unbounded oscillating term with very small coefficient due to zeros of Riemann Zeta function (long range behavior).

Plot of $f(r)$



Generalizations

- **Moebius Series over Arithmetic Progressions:**

$$\sum_{\substack{n=1 \\ n\equiv a \bmod q}}^{\infty} \mu(n),\ a, q \text{ positive integers}$$

- **Moebius Series with Coprimality Condition:**

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \mu(n),\ q \text{ positive integer}$$

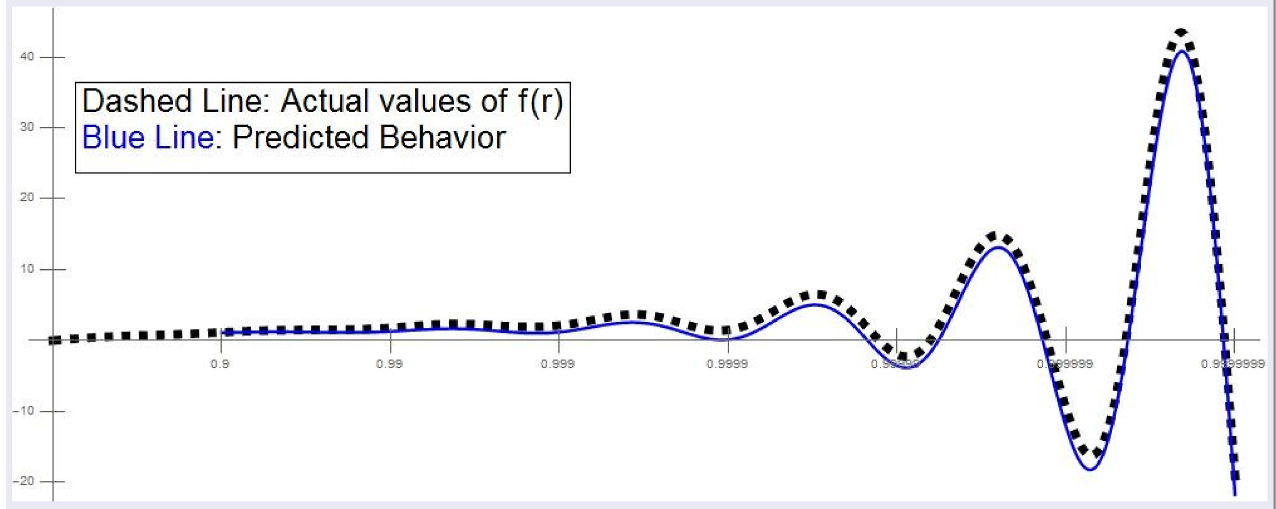
- **Moebius Series with Dirichlet Character:**

$$\sum_{n=1}^{\infty} \chi(n)\mu(n),\ q \text{ positive integer}$$

The generalizations are not Abel summable and Lambert summable (Delange, 2000). Like the original Moebius series, these generalizations have a similar predicted and numerical behavior.

Observing the Predicted Oscillations

- For the original Moebius series the predicted oscillations appear far beyond the computable range and thus cannot be verified.
- For certain Moebius Series over arithmetic progressions these oscillations appear earlier, and can be computationally tested.
- **Example:** The Moebius series over the arithmetic progression 1 mod 7. The following graph shows the actual behavior of the corresponding function $f(r)$ (dashed line) and the oscillations predicted by the analytic expansion of $f(r)$ (blue line).



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