Almost Beatty Partitions

Matthew Cho, Xiaomin Li, Yun Xie, Junxian Li (Graduate Student), A.J. Hildebrand (Faculty Advisor)

ILLINOIS   Illinois Geometry Lab

University of Illinois Undergraduate Research Symposium, April 19, 2018

Beatty Sequences

Definition

Let γ be an irrational number. The Beatty sequence B_γ is defined as

$$B_\gamma = \{\lfloor n\gamma \rfloor, n = 1, 2 \dots, \},$$

where $\lfloor x \rfloor$ is the floor function.

Example: Partition of Integers

Let ϕ be the golden ratio.

n	1	2	3	4	5	6	7	8
$n\phi$	1.62	3.24	4.85	6.47	8.09	9.70	11.33	12.94
$\lfloor n\phi \rfloor$	1	3	4	6	8	9	11	12
$n\phi^2$	2.61	5.24	7.85	10.47	13.09	15.71	18.33	20.94
$\lfloor n\phi^2 \rfloor$	2	5	7	10	13	15	18	20

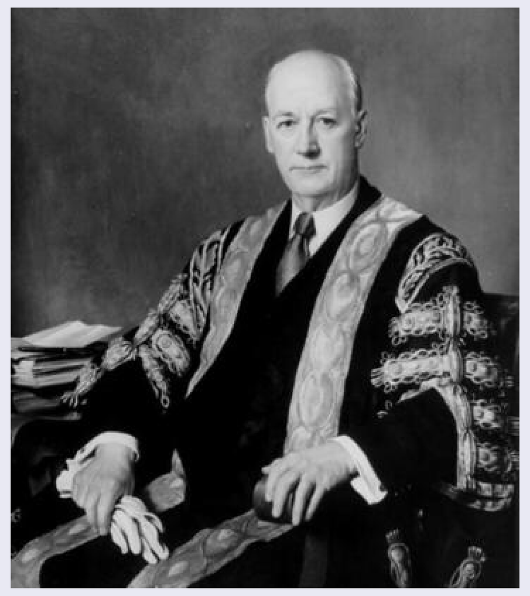
Beatty sequences of B_ϕ and B_{ϕ^2}.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Beatty Partition with B_ϕ and B_{ϕ^2}.

Samuel Beatty

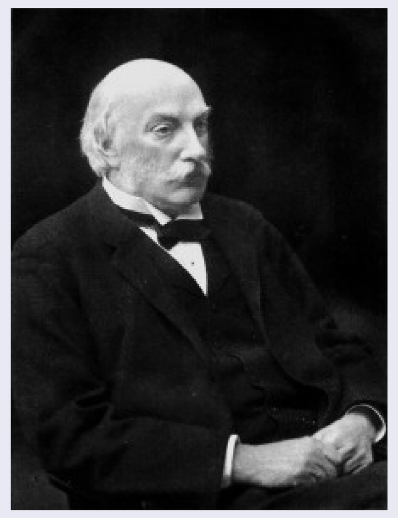
- Published Beatty's Theorem as a problem in the American Mathematical Monthly in 1926.
- First person receiving a Ph.D degree in mathematics from a Canadian university.
- One of the founders and the first president of the Canadian Mathematical Congress.



Samuel Beatty (1881-1970)

John W. Strutt (3rd Baron Rayleigh)

- Stated Beatty's Theorem even earlier in his book "The Theory of Sound" in 1894.
- Received the Nobel Prize in Physics in 1904 for discovering Argon.



John William Strutt (1842-1919)

Partitions of the Integers with Beatty Sequences

Theorem (Beatty's Theorem)

Let α and β be two positive irrational numbers. B_α and B_β form a partition of the integers if and only if

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Theorem (Uspensky's Theorem)

*Beatty's Theorem does not hold for three (or more) sequences. That is, if α, β and γ are arbitrary positive numbers, then B_α, B_β and B_γ do **not** partition the positive integers.*

How close can a 3-part partition be to three Beatty Sequences?

Theorem (3-part Almost Beatty Construction 1)

Let α and β be two irrational numbers such that B_α and B_β are disjoint. Let γ be the irrational number such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Denote

$$B_\gamma^* = \mathbb{N} \backslash (B_\alpha \cup B_\beta).$$

Let $B_\gamma(n)$ be the n-th term of B_γ. Then

$$\max_n (B_\gamma^*(n) - B_\gamma(n)) = \max\left(\left\lfloor \frac{1}{\alpha - 1} \right\rfloor, \left\lfloor \frac{1}{\beta - 1} \right\rfloor\right) + 2.$$

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

$\alpha = \phi^2$, (red) $\beta = \phi^3$, (blue) $B_\gamma^*(n) - B_\gamma(n) \in \{1, 2\}$.

Theorem (3-part Almost Beatty Construction 2)

Given $\alpha > 2$, let

$$B_\alpha^* = B_\alpha - 1.$$

Let γ be the irrational number such that

$$\frac{1}{\alpha} + \frac{1}{\alpha} + \frac{1}{\gamma} = 1.$$

Denote

$$B_\gamma^* = \mathbb{N} \backslash (B_\alpha \cup B_\alpha^*).$$

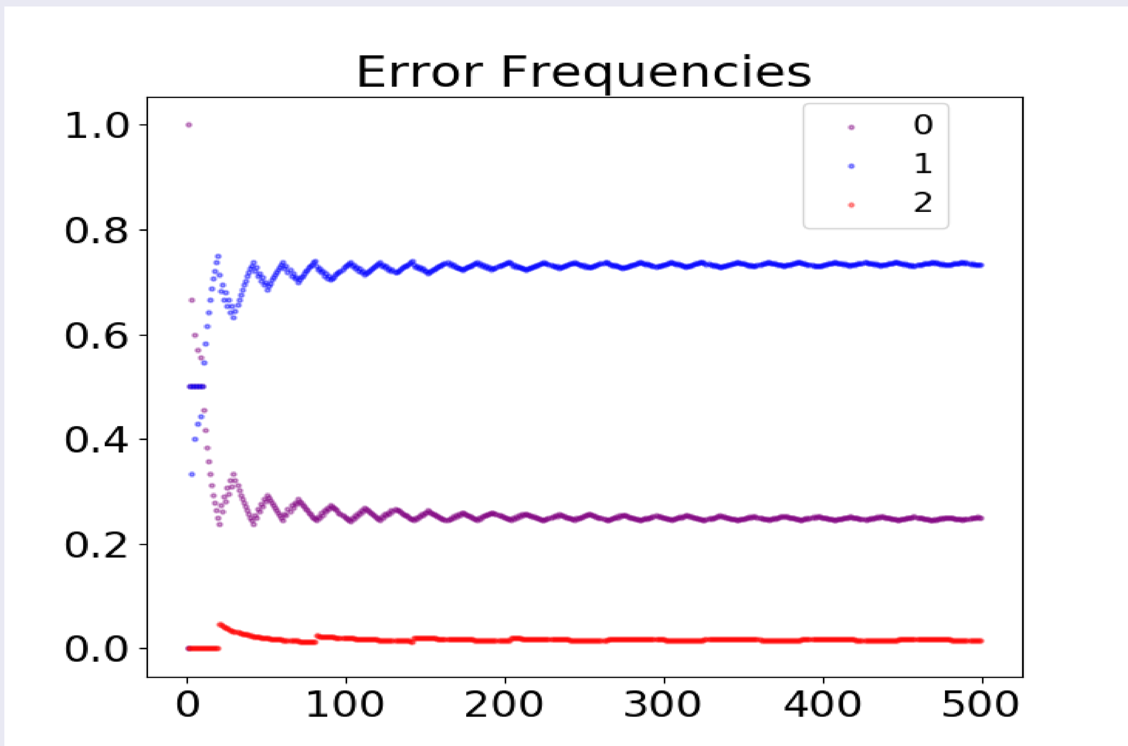
Let $B_\gamma(n)$ be the n-th term of B_γ. Then

$$B_\gamma^*(n) - B_\gamma(n) \in \{0, 1\}$$

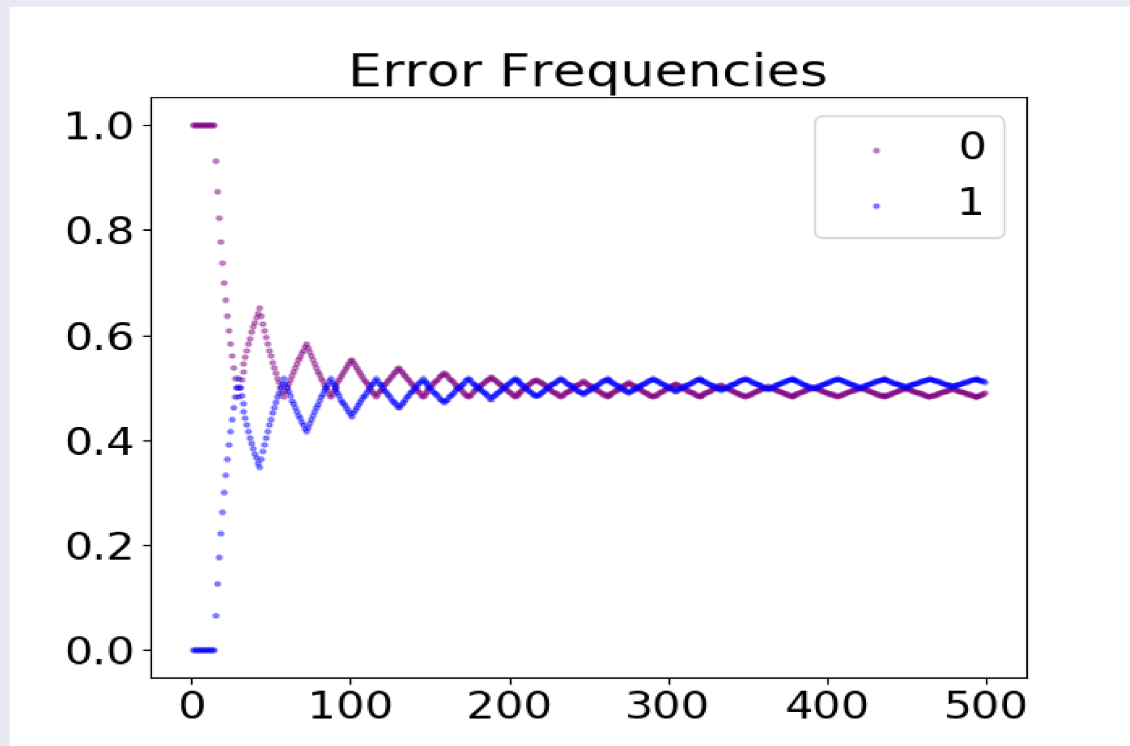
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

$\alpha = \phi^2$, (red) $B_\gamma^*(n) - B_\gamma(n) \in \{0, 1\}$.

Numerical Data on Distribution of Errors: $B_\gamma^*(n) - B_\gamma(n)$



$\alpha = \pi^3$, $\beta = \frac{3\alpha}{\alpha - 2}$, $B_\gamma^*(n) - B_\gamma(n) \in \{0, 1, 2\}$



$\alpha = \pi^3$, $B_\gamma^*(n) - B_\gamma(n) \in \{0, 1\}$,

Greedy Construction

Let α, β, γ be positive irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Construct sequences $B_\alpha^*, B_\beta^*, B_\gamma^*$ iteratively as follows:

- For each $n = 1, 2, 3, \dots$ place n into the sequence for which the error

$$B_\alpha^*(n) - B_\alpha(n)|, |B_\beta^*(n) - B_\beta(n)|, |B_\gamma^*(n) - B_\gamma(n)|$$

  is smallest.
- By construction, the resulting sequences $B_\alpha^*, B_\beta^*, B_\gamma^*$ form a partition of the positive integers.

Conjecture

The partition generated by the Greedy Construction satisfies

$$|B_\alpha^*(n) - B_\alpha(n)|, |B_\beta^*(n) - B_\beta(n)|, |B_\gamma^*(n) - B_\gamma(n)| \leq 2$$

for all positive integers n.

Numerical Example: Distribution of Errors

$\alpha \approx 2.27$, $\beta \approx 12.53$, $\gamma \approx 2.09$

Future Work

- Determine the possible errors and their frequencies in terms of α and β.
- Classify the cases when the errors are 0.
- Investigate the possible relations between different types of constructions.
- Extend current constructions to partitions with more than three parts.

References

- John William Strutt, 3rd Baron Rayleigh (1894). *The Theory of Sound*. 1 (Second ed.). Macmillan. p. 123.
- Beatty, Samuel (1926). *Problem 3173*. American Mathematical Monthly. 33 (3): 159. doi:10.2307/2300153
- Uspensky, J. V. (1927). *On a problem arising out of the theory of a certain game*. Amer. Math. Monthly 34 (1927), pp. 516–521.