Number-Theoretic Random Walks
Project Report

Yiwang Chen, Wenmian Hua, Natawut Monaikul, Tong Zhang
M.Tip Phaovibul (Team Leader)
A.J. Hildebrand (Faculty Mentor)

May 2, 2013

1 Introduction

Do numbers play dice? In many ways, the answer seems to be yes. For example, whether an integer has an *even* or *odd* number of factors in its prime factorization is a property that behaves much like the outcome of a coin toss: The even/odd sequence obtained, say, from a stretch of one hundred consecutive integers, is, at least on the surface, indistinguishable from the heads/tails sequence obtained by tossing a coin one hundred times.

We may thus think of the "even or odd number of prime factors" property as a "digital coin flip", encoded by a function $f(n)$ that takes on the value $+1$ if n has an even number of prime factors and -1 if n has an odd number of prime factors.[1]

Many functions in number theory exhibit such random-like behaviors and can thus serve as a "digital coin flip function". Our project is part of an ongoing project aimed at exploring the (non)-randomness of such functions *geometrically* by studying certain "random walks" in the plane formed with these functions. Such random walks provide a natural way to visualize and quantify the degree of randomness of these sequences. They can help detect and explain hidden patterns, but can also reveal new phenomena that have yet to be explained.

2 Number-Theoretic Random Walks

Given a digital coin flip function $f(n)$ with values ± 1 and a real number α, we consider sums of the form

$$S(N) = S(N, f, \alpha) = \sum_{n=0}^{N-1} f(n) e^{i\alpha n}, \tag{1}$$

These sums have a natural geometric interpretation as a random walk in the complex plane whose nth step is given by $f(n)e^{i\alpha n}$. We call such a random walk a number-theoretic random walk.

More intuitively, this random walk can be constructed as follows:

[1] This function $f(n)$ is known as the Liouville function, and it has been extensively studied. In particular, the degree of "randomness" in its behavior is closely related to the *Riemann Hypothesis*, one of the seven "Million Dollar Problems" in mathematics.

- Start at the origin $(0,0)$.
- For the initial step, move in the horizontal direction by one unit, either to the right $(+)$ or to the left $(-)$, depending on the outcome of the first digital coin flip, $f(0)$.
- At each subsequent step, change the direction by a fixed "turn angle" α, so that the direction after n steps is $n\alpha$.
- Move forward $(+)$ or backwards $(-)$ in this direction, depending on the outcome of the digital coin flip, i.e., the value of the function at the nth step, $f(n)$.

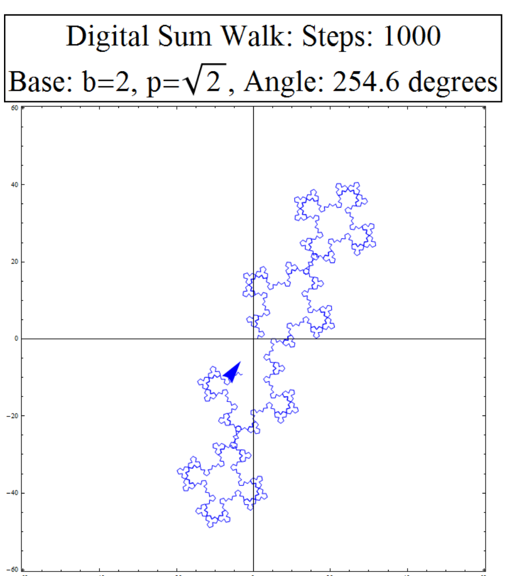
3 Digital Sum Random Walks

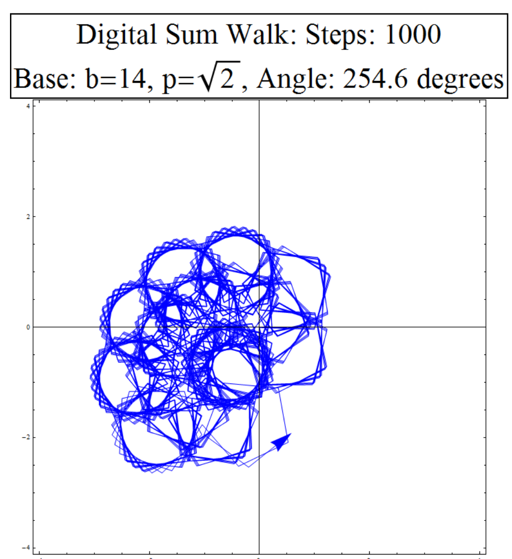
For the Spring 2013 project, we focused on a particular number-theoretic random walk, the "Digital Sum Random Walk" (DSRW), in which the digital coin flip function $f(n)$ is defined as follows: Fix an integer base b, and define $f(n)$ as $+1$ if n has an even sum of digits when expanded in base b and -1 otherwise. In other words, $f(n) = (-1)^{s_b(n)}$, where $s_b(n)$ is the sum of the digits of n in base b.

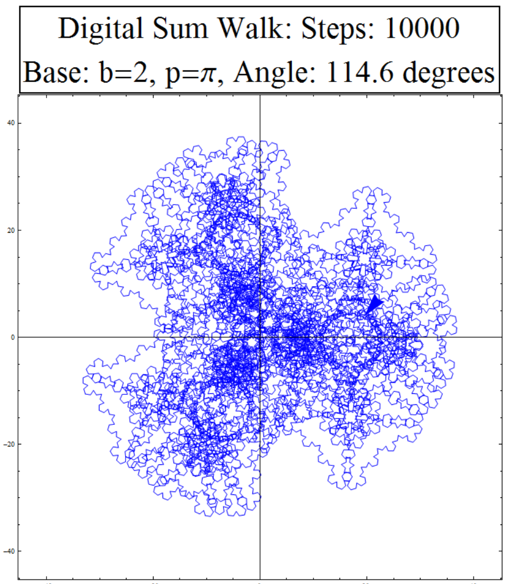
More formally, writing the turn angle α in the form $\alpha = 2\pi/p$, where p is a positive real number, we define the Digital Sum Random Walk with base b and parameter p by

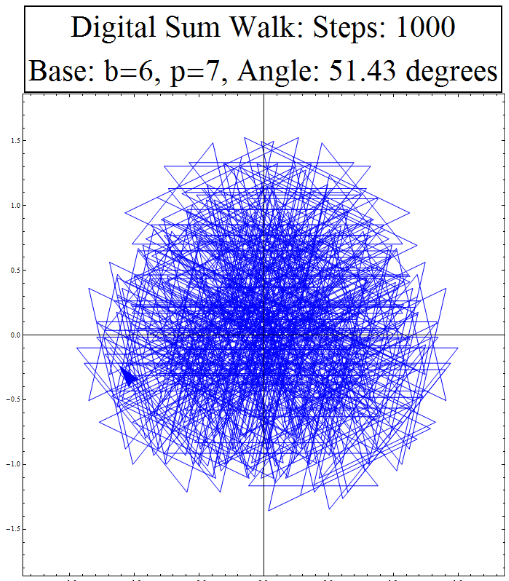
$$S(N) = S(N, b, p) = \sum_{n=0}^{N-1} (-1)^{s_b(n)} e^{2\pi i n/p}. \tag{2}$$

DSRWs exhibit a great variety of patterns and behaviors—much of which remains to be fully explained—and thus provide a fertile ground for further investigation. The figures below show some examples of DSRWs.









4 Results

In order to quantify the randomness in a number-theoretic random walk, we introduce the notion of a **scaling exponent**, defined as follows:

Definition. *Given a random walk $S(N)$ in the plane as above, its **scaling exponent** μ is defined as*

$$\mu = \limsup_{N\to\infty} \frac{\log S^*(N)}{\log N}, \quad S^*(N) = \max_{n\le N} |S(N)|. \tag{3}$$

Roughly speaking, a random walk $S(N)$ with scaling exponent μ "spreads out" at a rate of N^{μ}. A random walk with maximal rate of growth (i.e., $S(N) \approx N$) has scaling exponent 1, a bounded walk has scaling exponent 0, while a "true" random walk (i.e., a random walk of the above type with the coin flip function $f(n)$ given by independent Bernoulli random variables) has scaling exponent 0.5. Thus, the scaling exponent μ can be viewed as a measure for the "degree of randomness" of a number-theoretic random walk. The table below gives some numerical approximations, μ_{approx}, for DSRWs with various parameters b and p. The approximations were obtained by evaluating the quantity under the limit in (3) at $N = 10^6$.

Base b	p	$S^*(10^6)$	$\mu_{\text{approx}} = \log S^*(10^6)/\log(10^6)$
2	3	54715.5	0.789685
2	5	3003.26	0.579599
2	9	67.4276	0.304806
2	$\sqrt{2}$	57.5374	0.293325
2	π	48.4322	0.280856
3	π	1.85082	0.0445605
3	5	1.23607	0.0153404
5	9	1.06418	0.00450236

On the theoretical front, we were able to obtain exact values for the scaling exponent of DSRWs in several special cases.

Theorem (Scaling Exponent of Digital Sum Random Walks)**.** *Let $\mu(b,p)$ denote the scaling exponent of the Digital Sum Random Walk with base b and parameter p. Then:*

- $\mu(2,3) = \frac{\log 3}{\log 4} = 0.792481\ldots$, $\mu(2,5) = \frac{\log 5}{\log 16} = 0.580482\ldots$, *and* $\mu(2,p) < 0.5$ *for all other integer values* $p > 5$.

- $\mu(b,p) = 0$ *for any* odd *base* b *and any real number* $p > 2$.

Thus, in all of the above cases, the Digital Sum Random Walk grows either significantly slower or significantly faster than a genuine random walk.

5 Related Work and Future Directions

The initial motivation for this work came from a 1979 paper titled "Picturesque exponential sums" by D.H. and Emma Lehmer [7] and its generalization by Apostol [4], who investigated the geometric behavior of sums similar to (2).

The work of Lehmer and Apostol formed the topic of a 2009 Summer REG (Research Experience for Graduate Students) project by Michael DiPasquale and Dan Schultz [5], who generalized some of Apostol's results.

The "digital coin flip function" of the DSRW, $f(n) = (-1)^{s_b(n)}$, is a generalization of the Thue-Morse sequence (which corresponds to the case $b = 2$) and has been thoroughly studied; see, for example, [2] and [3].

In the case of *integer* values of p, the behavior of the DSRW is related to the behavior of the sum-of-digit function $s_b(n)$ when n is restricted to arithmetic progressions modulo p. This behavior is highly nontrivial and has been the subject of some extensive investigations over the past two decades; see, for example, [6]. It is likely that some of the geometric features of the DSRWs we have observed in the cases of integer parameters p can be explained using the results of this deep work.

The scaling exponent $\mu(b,p)$ for integer values of p is related to the so-called Gelfond exponent, arising as the exponent of the estimates involving the distribution of $s_b(n)$ in arithmetic progressions; see [9].

In the special case of $b = 2$ and $p = 3$, the DSRW is a version of the Koch curve, a classical fractal curve; Ma and Holdener [8] have studied this case in detail. In many other cases, the DSRW also shows distinctive fractal-like features, but the above case seems to be the only one where the DSRW is a classically known fractal.

As Ma and Holdener have observed, random walks of this type can be interpreted as instances of "turtle graphics"; see [1].

The case when p is irrational yields the most interesting, yet least understood, types of behaviors; this is a motivating direction for future work.

References

[1] Abelson, Harold; diSessa, Andrea A. *Turtle geometry.* MIT Press, Cambridge, Mass., 1981.

[2] Allouche, Jeam-Paul; Shallit, Jeffrey. *The ubiquitous Prouhet-Thue-Morse sequence.* Sequences and their applications (Singapore, 1998), 1–16. Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, London, 1999.

[3] Allouche, Jeam-Paul; Shallit, Jeffrey. *Automatic sequences.* Cambridge Univ. Press, Cambridge, 2003.

[4] Apostol, Tom M. *An extension of the Lehmers' picturesque exponential sums.* Math. Comp. 61 (1993), no. 203, 25–28.

[5] DiPasquale, Michael; Schultz, Dan. *Extended results on Lehmers' picturesque sums.* REGS Report, 2009. `http://www.math.illinois.edu/REGS/reports/DiPasquale-Shultz.pdf`

[6] Drmota, Michael; Skalba, Mariusz. *Rarified sums of the Thue-Morse sequence.* Trans. Amer. Math. Soc. 352 (1999), 609–642.

[7] Lehmer, D. H; Lehmer, Emma. *Picturesque exponential sums. I.* Amer. Math. Monthly 86 (1979), no. 9, 725–733.

[8] Ma, Jun; Holdener, Judy. *When Thue-Morse meets Koch.* Fractals 13 (2005), no. 3, 191–206.

[9] Shparlinski, I. *On the size of the Gelfond exponent.* J. Number Theory 130 (2010), 1056–1060.