Interactive Visualizations in Calculus and Probability

Project Report, Spring 2019

Raymond Harpster, Tianli Li, Yikai Teng
A.J. Hildebrand (Faculty Mentor)

Illinois Geometry Lab
University of Illinois at Urbana-Champaign

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1 Introduction

This is the second semester of a long-term program aimed at creating interactive Mathematica-based visualizations of interesting mathematical topics and making these available to a broader audience through publication at the Wolfram Demonstrations website, `http://www.demonstrations.wolfram.com`. The ultimate goal is to develop a collection of attractive interactive tools for use in instruction and outreach activities.

This semester we focused on visualizations of topics in multivariable calculus and combinatorics. In future semesters we may branch out into other areas such as number theory, game theory, and differential equations.

2 The Geometry of Lagrange Multipliers

The method of Lagrange multipliers refers to a strategy for finding the local extrema of a function $f(x, y)$ under a constraint $g(x, y) = k$. The method is based on the observation that (under suitable smoothness conditions) the local extrema are points at which the gradient of the constraint function $g(x, y)$ is parallel to the gradient of the function $f(x, y)$ to be optimized. Thus, the local extrema can be found by solving the system of equations

$$\begin{aligned} \nabla f(x, y) &= \lambda \, \nabla g(x, y), \\ g(x, y) &= k. \end{aligned}$$

Visualizing the Lagrange Multiplier Method. The Wolfram Demonstration below illustrates the geometry behind the Lagrange Multiplier Method for over 150 combinations of surfaces $z = f(x, y)$ and constraint curves $g(x, y) = k$. After choosing a surface/curve combination, the user can move a point along the constraint curve using the built-in slider, and the Demonstration then displays the gradient vectors ∇f and ∇g (normalized to unit vectors), as well as the z-value of the surface $z = f(x, y)$, at this point. One can then observe that at local extrema the two gradient vectors are parallel.

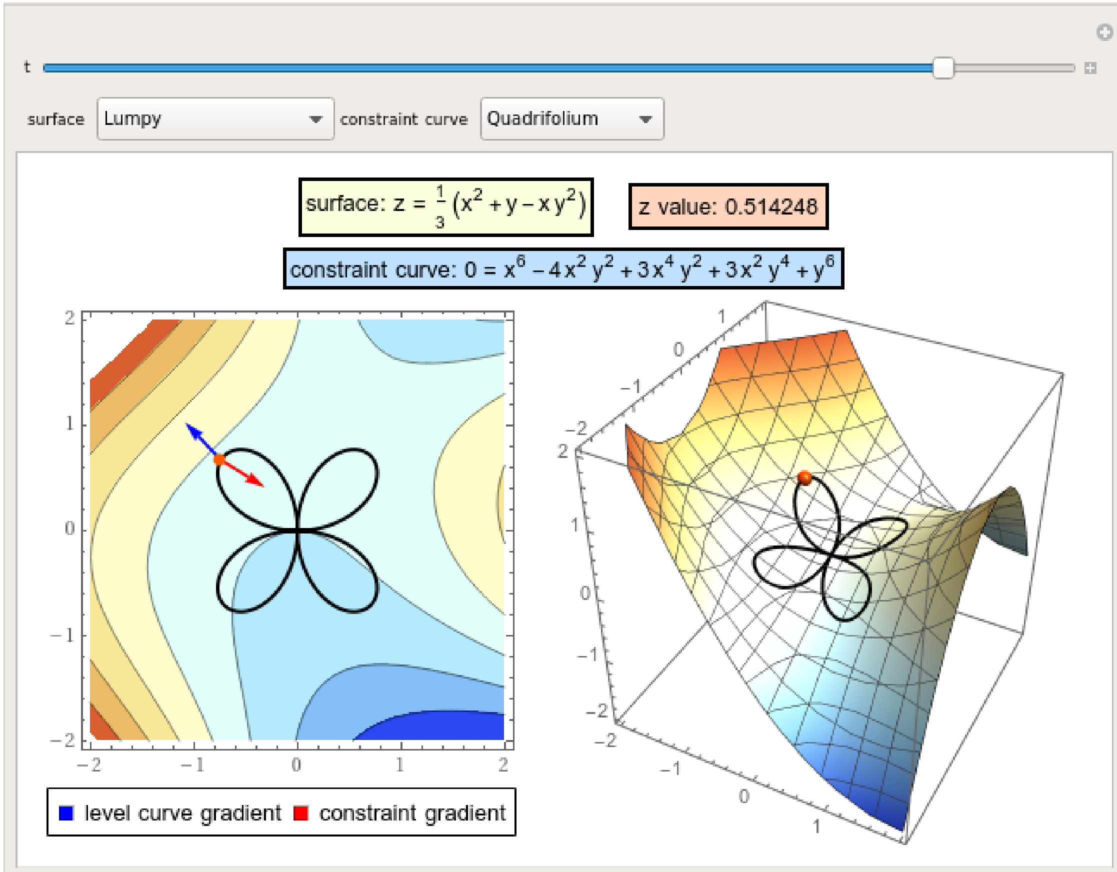

Figure 1: The Lagrange Multiplier Demonstration

3 The Coupon Collector Problem

The Coupon Collector Problem is a problem in combinatorial probability that can be stated as follows:

> *Suppose each box of cereals contains a coupon chosen at random from n possible coupons. How many boxes of cereals need to be purchased in order to get a complete set of all n coupons?*

More precisely, if X denotes the number of boxes that need to be purchased, the problem asks for:

(I) The expected value of X, i.e., $E(X)$.

(II) The distribution of X, i.e., the probabilities $P(X = k)$, for $k = 1, 2, 3, \ldots$.

The solution is given by the following formulas:

(1)
$$E(X) = n \cdot \sum_{i=1}^{n} \frac{1}{i}$$

(2)
$$P(X = k) = \frac{1}{n^{k-1}} \sum_{i=0}^{n-1} (-1)^i (n-1-i)^k \binom{n-1}{i}.$$

See, for example, [1] and [4].

Visualizing the Coupon Collector Problem. The Wolfram Demonstration below illustrates the coupon collector problem for the following natural sets of "coupons": The 10 digits $0, 1, \ldots, 9$; the 4 suits of a poker deck; the 13 cards in a single suit; and the 6 sides of a standard die. The Demonstration generates a random sequence of coupons from the selected set, it shows the number of coupons of each type obtained so far, and it stops as soon as a complete set of coupons has been obtained. The total number of coupons that have been collected is then compared with the expected value given by formula (1).

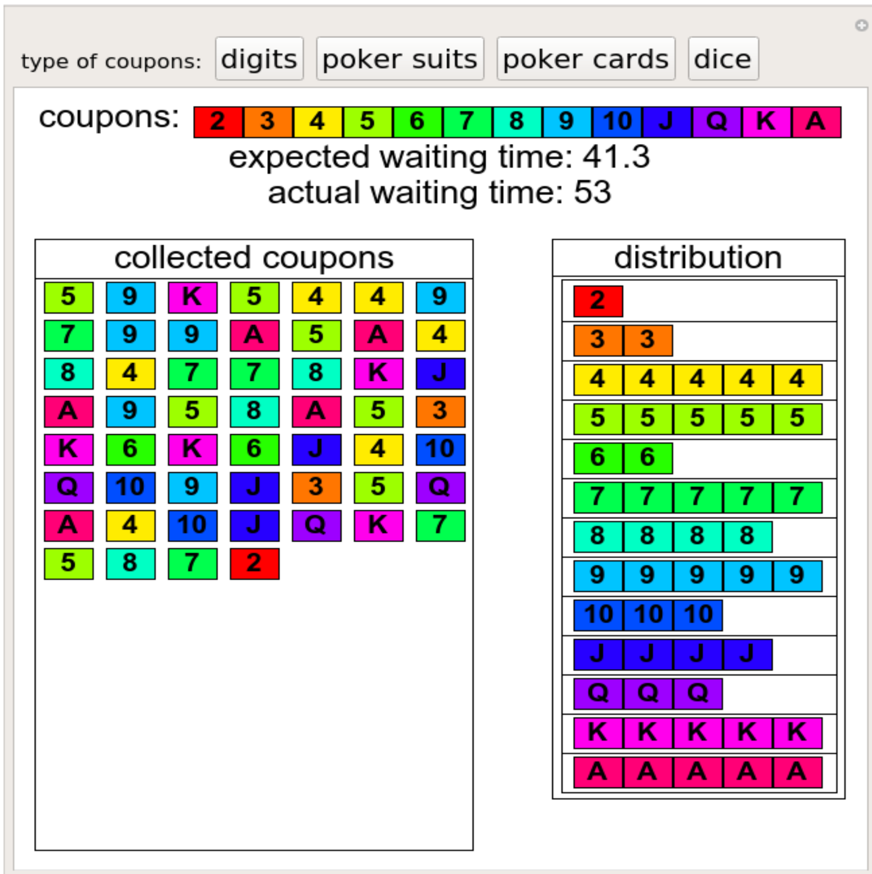

Figure 2: The Coupon Collector Demonstration

4 The Coupon Collector Randomness Test

The coupon collector randomness test [2], [3] is a randomness test for sequences based on the coupon collector problem. It can be applied, in particular, to test the randomness of digits of famous irrational numbers such as π.

The test works as follows: We consider the digits $0, 1, \ldots, 9$ as coupons and we compute the number of terms in the sequence needed to obtain a complete collection of all ten digits. This is the first "coupon waiting time", X_1. We repeat this process to obtain a sequence of coupon waiting times $X_1, X_2, X_3, \ldots$ and compare the mean and distribution of these *observed* coupon waiting times to the *theoretical* mean and distribution, given by the formulas (1) and (2). A good match suggests that the sequence of digits tested behaves like a random sequence, while a poor match is evidence that the sequence is not random.

Visualizing the Coupon Collector Randomness Test. This Demonstration illustrates the coupon collector randomness test for initial sequences of digits of famous irrational numbers, and some rational approximations of π. The observed waiting time frequencies are shown as a bar chart, while the theoretical frequencies are shown as a solid line. The Chi-square test, a standard statistical test to compare two discrete distributions, is used to measure how close the two distributions are. The test outputs a p-value that, very roughly, represents the likelihood that the sequence is indeed random.

One can observe that for rational numbers (including rational approximations to π) the p-values approach 0 as the number of terms in the sequence increases. This is consistent with the well-known fact that such numbers have periodic (and hence non-random) decimal expansions. On the other hand, for irrational numbers such as the number π, the p-values remain of moderate size no matter how many terms of the sequence are taken. This is consistent with the widely accepted (but yet unproven) belief that the digits of π behave like a random sequence.

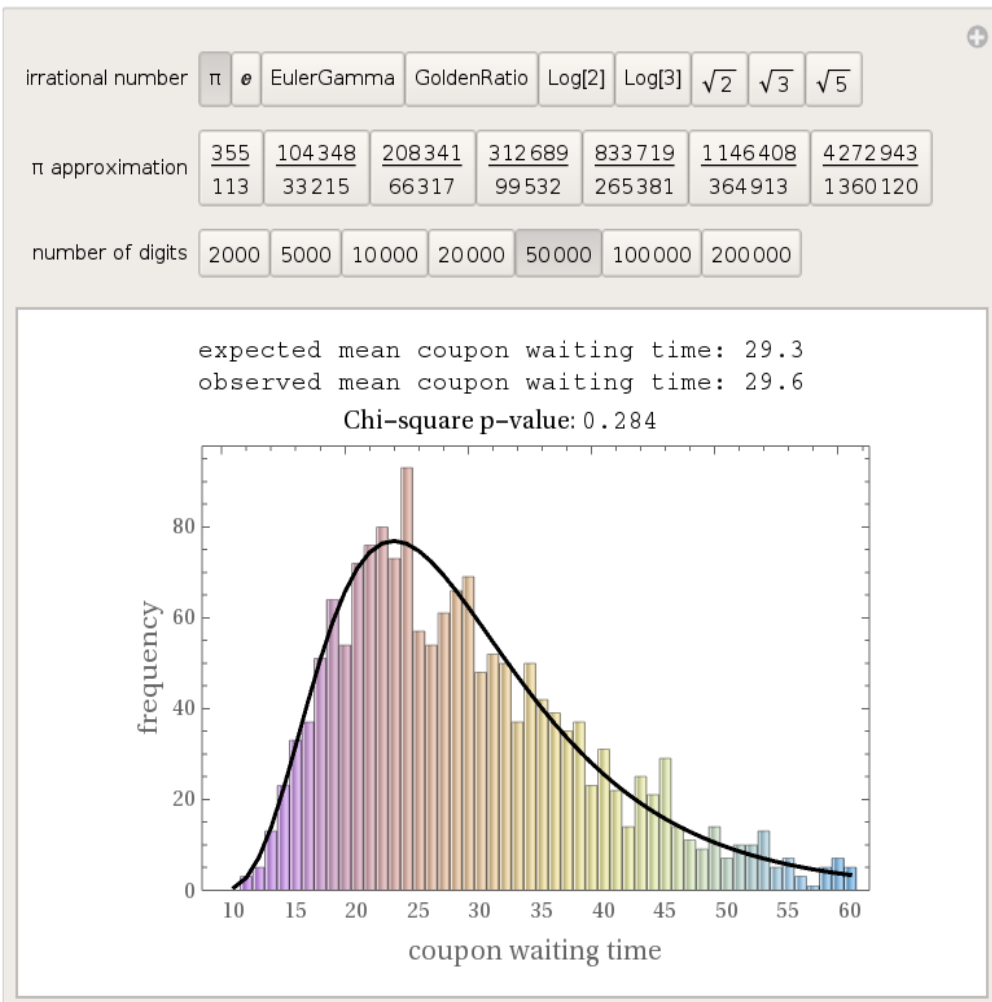

Figure 3: Demonstration of the Coupon Collector Randomness Test

References

[1] Dawkins, B. (1991). *Siobhan's problem: the coupon collector revisited.* The American Statistician, 45(1), 76-82.

[2] Greenwood, R. E. (1955). *Coupon collector's test for random digits.* Mathematical Tables and Other Aids to Computation, 1-5.

[3] Knuth, D. E. (2014). *Art of computer programming, volume 2: Seminumerical algorithms.* Addison-Wesley Professional.

[4] Von Schelling, H. (1954). *Coupon collecting for unequal probabilities.* The American Mathematical Monthly, 61(5), 306-311.