The Mathematics of Poker-like Games
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1 Introduction

This project is part of an ongoing program focusing on mathematical questions in game theory, and in particular on mathematical models of poker and poker-like games. This is an active area of research at the interface of mathematics and economics with a wide range of real-world applications. In past editions of this project we have investigated variations of the classic two-player poker model by John von Neumann [5, 6], a three-player poker model created by John Nash and Lloyd Shapley [4], and recently created poker models by David McAdams [1] and Noga Alon [2]. This semester we continued our investigations of McAdams and Alon type models.

2 The Poker Models of McAdams and Alon

The McAdams poker model [1] is a symmetric two-player poker model defined as follows:

1. Each player is dealt a "hand", represented by a random number drawn from the interval $[0, 1]$.
2. Each player sees his/her hand (but not the opposing player's hand) and decides to either **bet** or **fold**. The two betting actions occur simultaneously.
3. The two "hands" (i.e., random numbers in $[0, 1]$) are then revealed and the players are payed out according to the following rules:
   - If one player bets and the other folds, the player who bets wins \$1, while the other player loses \$1.
   - If both players bet, the player with the best hand (i.e., largest number) wins \$2, while the other player loses \$2.

- If both players fold, the player with the best hand (i.e., largest number) wins \$1, while the other player loses \$1.

The Alon model proceeds in a similar way except that the payout rules are different: If one or both players fold, no exchange of money occurs, while if both players bet, the player with the best hand wins \$1, while the other player loses \$1.

Table 1 illustrates the rules of the McAdams and Alon models. Here the $\pm$ sign depends on which player has the better hand: If Player I has the better hand, the plus sign holds, otherwise, the minus sign holds.

Player I	Player II	McAdams Model	Alon Model
Bet	Bet	± 2	± 1
Bet	Fold	1	0
Fold	Bet	-1	0
Fold	Fold	± 1	0

Table 1: Profit for Player I in the poker models of McAdams and Alon.

3 Extensions of the McAdams and Alon Models

In our project, we consider two types of extensions of the McAdams and Alon Models:

- **Discrete extension.** We replace the continuous range $[0,1]$ for the hand values by a finite discrete range of the form $\{1, 2, \ldots, n\}$. Thus there are n possible hands, each equally likely to be drawn. This is a more realistic model of poker and similar games involving cards drawn from a finite deck of cards.
  Handling of ties. In a poker model with a continuous range of hand values, "ties" between hands have probability 0 and thus do not affect expected profits. In the discrete case, however, ties can occur with positive probability, and the rules of the game need to account for the case of ties. The natural way to do this is by letting no money exchange hands if there is a tie in the bet/bet and fold/fold cases.

- **Continuous extension.** In this extension, the hands are drawn uniformly from the interval $[0, 1]$ as in the original models, but the payouts for the bet/fold, bet/bet, and fold/fold cases are replaced by general nonnegative values a, b, c, as shown in Table 2 below.

Player I	Player II	Generalized McAdams Model
Bet	Bet	$\pm b$
Bet	Fold	a
Fold	Bet	$-a$
Fold	Fold	$\pm c$

Table 2: Profit for Player I in the Generalized McAdams Model.

4 Strategies, Best Responses, and Nash Equilibria

A **strategy** in a McAdams or Alon type game is a recipe that, based on the value of a player's hand, tells the player whether to bet or to fold. For example, a player might decide to always bet with a "good" hand and always fold with a "bad" hand, where a hand is defined as "good" if its value is above a certain cutoff c. This is an example of a deterministic (or "pure") strategy.

Another important class of strategies are **probabilistic strategies**, where the decision whether to bet or fold depends not only on the player's hand value, but also on some random event. For example, a player might decide by a coin toss whether or not to bet if the hand value is in a certain range. Such a probabilistic strategy introduces an element of unpredictability into the game, in the sense that it keeps the opponent guessing.

In our work, we consider a two parameter family (c, p) of probabilistic strategies defined as follows:

- If the player's hand value is $> c$, bet always.
- If the player's hand value is $\leq c$, bet with probability p.

Here c and p are arbitrary parameters in $[0, 1]$. The parameter p represents the probability of betting with a "bad" hand and thus can be interpreted as a **bluffing probability**. The case $p = 0$ corresponds to the above simple strategy of always betting if the hand value is above a cutoff, and always folding if the value is below this cutoff. Figure 1 shows the betting probability for this strategy as a function of the hand value.

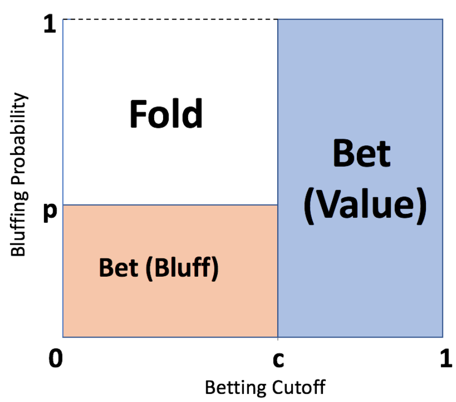

Figure 1: Strategy (c, p).

A **Nash Equilibrium** is a pair of strategies—one for each player—such that no player can increase his/her expected profit by changing to a different strategy. McAdams [1] showed that the strategy $(1/2, 1/3)$ (i.e., "always bet if the hand value is $> 1/2$, bet with probability $1/3$ if the hand value is $\leq 1/2$") represents a symmetric Nash Equilibrium for his poker model. That is, if both players follow this strategy, then no player can gain an advantage.

Given a strategy for one player, a **best response strategy** is a strategy for the opposing player that maximizes this player's expected profit.

A **best response iteration** is a sequence of strategies, each of which is the best response to the preceding strategy. Such a sequence can lead to single a fixed point, given by a Nash equilibrium, or to a **best response cycle**.

5 Summary of Results and Future Work

For the continuous and discrete extensions of the McAdams and Alon models defined above we obtained the following results:

- **Nash equilibria.** We showed that there is no Nash equilibrium strategy of the form $(c, 0)$. In other words, any strategy of the form "bet if hand value is $> c$ and fold if the hand value is $\leq c$" can be beaten by a counter-strategy of the same form (with a different c-value).

  On the other hand, we also showed that there does exist a Nash equilibrium strategy of the form (c, p), and we determined this strategy explicitly in terms of the betting parameters in Table 2.

- **Best response strategies.** We determined the best response strategies to strategies of the form $(c, 0)$, and the expected profit for each player under these strategies.

- **Best response cycles.** We showed that iteration of the best response strategy starting with a strategy of the form $(c, 0)$ always leads to a cycle, and we determined these cycles explicitly for some special cases.

Possible directions for future work include extending our results to strategies of the form (c, p), investigating the lengths of the best response cycles and their dependence on the parameters of the model, and analyzing more closely situations in which there is no unique best response strategy.

References

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