Intersecting Cylinders: From Archimedes and Zu Chongzhi to Steinmetz and Beyond
Project Report*

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1 Background

Our work was motivated by the following problem, which can be found in many calculus texts as a challenging example of computing volumes by multiple integrals.

Intersecting Cylinders Problem. *What is the volume of the region of intersection of two or three pairwise perpendicular cylinders of unit radius ("Steinmetz solids")?*

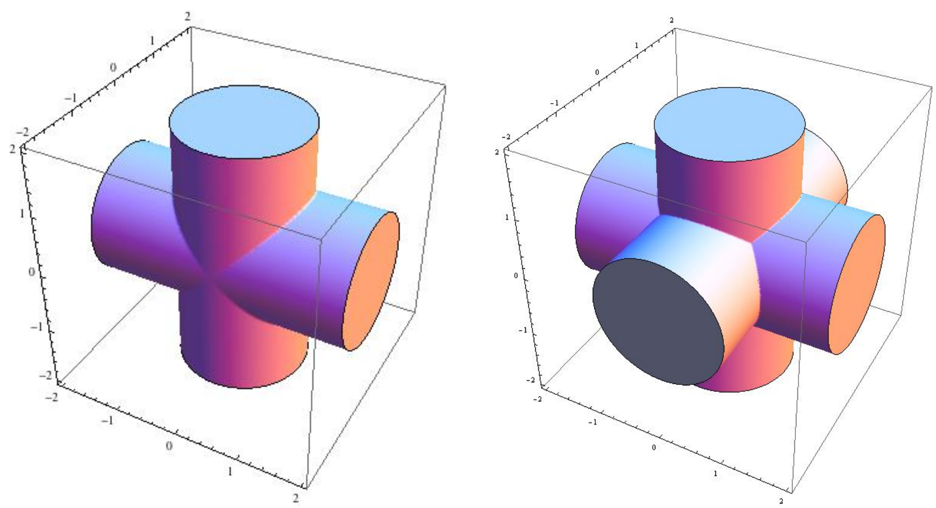

Figure 1: The classical 2-cylinder and 3-cylinder Steinmetz solids.

*This project is part of an ongoing research project "Adventures with n-dimensional integrals", carried out at the Illinois Geometry Lab, `www.math.illinois.edu/igl`. An expanded version of this report is expected to be prepared for possible publication.

The above solids are named after Charles Proteus Steinmetz (1865–1923), a brilliant electrical engineer and mathematician who is said to solved the two-cylinder version of this problem in two minutes at a dinner party.

The two-cylinder version of the problem was first studied more than two thousand years by Archimedes and the Chinese mathematician Zu Chongzhi, who proved that the volume of a two-cylinder Steinmetz solid is 16/3. The three-cylinder version was first considered in the 1970s by Moore [3], a physicist who was motivated by applications in crystallography. Using calculus methods, Moore showed that the volume of a three-cylinder Steinmetz solid is $16 - 8\sqrt{2}$.

The problem has been popularized by Martin Gardner in his Scientific American column [1], and the two-dimensional Steinmetz solid is featured on the cover of one of Gardner's books [2]. It received further notoriety when Steven Strogatz mentioned it in his *New York Times* column [5] as an illustration of Archimedes' slicing method to compute volumes. The problem has its own MathWorld entry [6] and Wikipedia page [7], where more information, and many references, can be found.

Our goal is to formulate, and solve, generalizations of this problem to higher dimensions.

2 Intersecting Cylinders in Higher Dimensions

In order to come up with an appropriate n-dimensional analog of the Steinmetz solid, we first need to introduce a notion of a cylinder in n-dimensional space. In the usual three-dimensional space $\mathbb{R}^3$, a cylinder of radius r about an axis L can be described as the set of all points in $\mathbb{R}^3$ whose distance to L is at most r.

We now generalize this definition to the n-dimensional space $\mathbb{R}^n$.

Definition. *Given a line L in the n-dimensional space $\mathbb{R}^n$ and a positive number r, we define the* ***cylinder of radius** r **about the axis*** *L as the set of all points in $\mathbb{R}^n$ whose distance to L is at most r.*

In particular, the cylinders of radius 1 centered at the n coordinate axes in $\mathbb{R}^n$ are given by the following equations:

$$\text{(1)} \qquad x_2^2 + x_3^2 + \cdots + x_n^2 \le 1 \quad \text{(cylinder about } x_1\text{-axis)},$$

$$\text{(2)} \qquad x_1^2 + x_3^2 + \cdots + x_n^2 \le 1 \quad \text{(cylinder about } x_1\text{-axis)},$$

$$\cdots$$

$$\text{(n)} \qquad x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \le 1 \quad \text{(cylinder about } x_n\text{-axis)}.$$

Having defined n-dimensional cylinders, we now generalize the definition of a Steinmetz solid to n dimensions as follows:

Definition. *Let m and n be integers with $2 \le m \le n$.*

- *The* ***n-dimensional Steinmetz solid****, S_n, is the intersection of* ***all*** *n cylinders (1)–(n).*
- *The* ***m-cylinder, n-dimensional Steinmetz solid*** *$S_{m,n}$ is the intersection of* ***exactly*** *m of the cylinders (1)–(n). In particular, $S_{n,n} = S_n$.*

Note that $S_{2,3}$ and $S_{3,3}$ are the classical two-cylinder and three-cylinder Steinmetz solids depicted in Figure 1.

3 Results

We are interested in computing the volumes, $\mathrm{Vol}(S_n)$ and $\mathrm{Vol}(S_{m,n})$, of the generalized Steinmetz solids defined above. This is the n-dimensional analog of the problem stated at the beginning.

Our first result gives an exact formula for the volume of S_n for dimensions 4 and 5:

Theorem 1 (Volume of S_4 and S_5)**.**

$$\mathrm{Vol}(S_{4,4}) = 48\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\right), \quad \mathrm{Vol}(S_5) = 256\left(\frac{\pi}{12} - \frac{1}{\sqrt{2}}\cot^{-1}(2\sqrt{2})\right).$$

Our second result is a "Lifting Theorem" that relates the volume of S_m to that of $S_{m,n}$ for any dimension $n \geq m$:

Theorem 2 (Lifting Theorem)**.**

$$\mathrm{Vol}(S_{m,n}) = \frac{\mathrm{Vol}(H_n}{\mathrm{Vol}(H_m)}\,\mathrm{Vol}(S_m),$$

where H_i denotes the i-dimensional (solid) hypersphere of radius 1.

Using known formulas for $\mathrm{Vol}(H_i)$, this result thus reduces the computation of $S_{m,n}$ to that of S_n.

Our theoretical results are consistent with numerical calculations of these volumes using Monte Carlo methods.

References

[1] M. Gardner, Scientific American 207(5) (1962), 164.

[2] M. Gardner, *The unexpected hanging and other mathematical diversions*, Chicago University Press, Chicago, IL, 1991.

[3] M. Moore, *Symmetrical Intersections of Right Circular Cylinders*, Math. Gazette 58 (1974), 181–185.

[4] J. Stewart, *Calculus*, 7th Edition, Brooks Cole, 2011

[5] S. Strogatz, *It slices, it dices*, New York Times, April 18, 2010, `http://opinionator.blogs.nytimes.com/2010/04/18/it-slices-it-dices`

[6] E. Weisstein, *Steinmetz Solid*, MathWorld, `http://mathworld.wolfram.com/SteinmetzSolid.html`.

[7] *Wikipedia Article on Steinmetz Solid*, `http://en.wikipedia.org/wiki/Steinmetz_solid`.