Calculus, Geometry, and Probability in n Dimensions: The Nash-Shapley Model for Multi-player Poker Games

Project Report, Spring 2018

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1 Introduction

This project is part of an ongoing program, begun in Fall 2012, aimed at seeking out and exploring interesting problems at the interface of calculus, geometry, and probability that are accessible at the calculus level, but rarely covered in standard calculus courses. Such problems are often motivated by natural questions arising in probability, statistics, economics, and other areas.

In recent semesters the focus of this project has been on topics at the interface of mathematics and economics, and specifically on mathematical and game-theoretic aspects of poker and poker-like games. In Fall 2016 and Spring 2017 we investigated the classical two-player poker model of John von Neumann [5] and its extensions and variations due to Bellman and Blackwell [1], Ferguson and Ferguson [2], and others. These models provide mathematical (rather than psychological) justifications for using bluffing as part of an optimal strategy

In Fall 2017 we have turned our attention to poker models involving more than two players. In contrast to two-player models, which have been extensively investigated in the literature and are now fairly well understood, models involving more than two players are still very poorly understood. This is in large part because the complications arising when incorporating a third (and fourth, etc.) player into the classical two-player models are such that a complete game-theoretic analysis appears to be out of reach.

2 The Nash-Shapley Poker Model

In 1950 John Nash and Lloyd Shapley [4] proposed, and completely analyzed, a simplified three-player poker model that involved only two types of cards, a "high" card and a "low" card. Even under these drastically simplified assumptions, the analysis of the game proved to be extraordinarily complex: In order to find the optimal strategy for each of the three players, Nash and Shapley set up and solved a system of equations involving 24 variables, each representing a betting probability under a particular scenario. Many of the details in the calculations were left out of the paper, thus making it a challenge to verify and replicate their results, let alone to extend and generalize their work.

Remarkably, Nash and Shapley did all of their analysis by hand, guided by the deep mathematical intuition that they both possessed[1]. What they have accomplished with their paper is a feat that few researchers of the time would have been able to perform. To this date, more than sixty years after its original publication, the Nash-Shapley three-player model remains the only model in the literature involving more than two players that has been completely analyzed.

3 Project Goals

In this project we seek to use modern tools such as *Mathematica* to:

(I) Verify and replicate the analysis of Nash and Shapley.

(II) Extend and generalize the Nash-Shapley model in a variety of directions.

The first goal serves as a test and training case for the capabilities of *Mathematica* and as an independent verification of the results reported in the paper of Nash and Shapley. Such a verification in itself would be a worthwhile endeavor as it may well be the case that the details of the Nash-Shapley analysis have never been checked independently, given extraordinarily complex nature of the calculations required.

The second goal is more ambitious and would break new ground in the almost completely unexplored landscape of multi-player poker models.

In Fall 2017 we completed the first of the above two goals: We have implemented the Nash-Shapley game as a symbolic optimization problem in *Mathematica*, and we developed the necessary code to determine optimal strategies in this game. The results obtained with *Mathematica* are in full agreement with those reported by Nash and Shapley.

In Spring 2018 we started focusing on the second goal, exploring generalizations and extensions of the Nash-Shapley model. Specifically, we investigated the effect of suboptimal play by one or two of the players.

4 The Nash-Shapley Model with Suboptimal Players

The Nash-Shapley analysis was based on the assumption that all three players are perfectly skilled and play rationally in the sense of trying to maximize their profits. Under this assumption, Nash and Shapley derived the optimal strategy for the each of three players. The set of strategies obtained in this way is called the *Nash Equilibrium* of the game. It is optimal in the sense that if any of the players deviates from this strategy, then that player's expected profit decreases. Thus, assuming all players are perfectly skilled and play rationally, no player has an incentive to change from the Nash Equilibrium strategy to a different strategy.

The situation is completely different when one of the players is known to be unskilled or semi-skilled. Indeed, according to a well-known phenomenon in game theory the best response strategy to an unskilled player is, in general, *not* the Nash Equilibrium strategy, but rather a different strategy that is effective against a specific type of unskilled player, but would, in general, not be effective when playing against a perfectly skilled player or other types of unskilled players.

In Spring 2018 we started to systematically analyze the effects of different types of suboptimal play by one or two of the players. The paper by Larkey et al. [3], who had performed such an

[1]Both Nash and Shapley would go on to receive Nobel prizes in Economics. John Nash is the subject of the book (and movie) "A beautiful mind".

analysis on two-player poker models, served as guide and model for our work. Similar to [3] we introduce the following player profiles:

- **Optimal Player:** A player who always plays the Nash Equilibrium strategy.
- **Naive Player:** A player who always bets on high cards and always passes on low cards.
- **Random Player:** A player who bets or passes with equal probability, regardless of the value of his or her card or the actions of other players.
- **Loose Player:** A player who always bets when holding a high card and bets 50% of the time when given a low card.
- **Tight Player:** A player who bets 50% of the time when holding a high card and never bets with a low card.

In our project we determined, theoretically and experimentally, the expected profit and the optimal response strategy for each player, given the player profiles of the two opponents. Here are some or our preliminary findings:

(1) The "optimal" strategy (optimal in the sense of the Nash Equilibrium) is virtually never the best response when one of the opponents is a suboptimal player.

(2) Among the five player profiles defined above, the "naive" player comes out ahead in almost all cases.

(3) The results of Nash and Shapley show that, if all three players play optimally, then Player 3 has a significant advantage over Players 1 and 2, and is the only player with a positive expected profit. Surprisingly, the opposite is the case with some of the other strategies. In particular, if all three players are "Random Players", then Player 1 has the edge and is the only one with a positive profit.

References

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