Calculus, Geometry, and Probability in n Dimensions: The Mathematics of Poker

Project Report, Spring 2017

Nicholas Brown, Junghyun Hwang, Ki Wang, Ruoyu Zhu
A.J. Hildebrand (Faculty Mentor)

Illinois Geometry Lab
University of Illinois at Urbana-Champaign

May 9, 2017

Contents

1 Introduction **2**

2 Project Goals **2**

3 The von Neumann Poker Model **2**

4 The Indian Poker Model **4**

5 Data Mining a Poker Database **5**

6 Future Directions **6**

1 Introduction

This project is part of an ongoing program, begun in Fall 2012, aimed at seeking out and exploring interesting problems at the interface of calculus, geometry, and probability that are accessible at the calculus level, but rarely covered in standard calculus courses. Such problems are often motivated by natural questions arising in probability, statistics, economics, and other areas.

This year we are focusing on mathematical models of poker developed by John von Neumann and others. These models give optimal strategies for certain simplified versions of poker, and they provide mathematical justifications for bluffing.

2 Project Goals

The general goals of this project are to:

- Learn selected topics from game theory, in particular, the poker models due to John von Neumann [8, 9], D.J. Newman [10], and others [1, 2, 3, 4, 5, 6, 11].
- Develop visualizations and interactive *Mathematica* demonstrations that illustrate these poker models, and make these tools widely available for educational purposes and outreach activities, for example, through the *Wolfram Demonstrations Project*.
- Develop models for variations of the standard poker game, such as "Indian Poker", and determine optimal strategies for these models.
- Analyze real-world poker data from the *University of Alberta IRC Poker Database* [7] using statistical and machine-learning tools.
- Present this work to mathematical audiences, the campus community, and the broader public through conferences, open houses, and outreach events.

3 The von Neumann Poker Model

The basic poker model created by John von Neumann [8, 9] is a simplified poker game involving two players, Player I and Player II. At the beginning of the game, each player pays an *ante* of 1 unit to the *pot*, so the pot has an initial value of 2 units. Then the two players are each dealt a poker hand, represented by random numbers x and y in the interval $[0, 1]$, where x denotes the value of Player I's hand, and y the value of Player II's hand. The value of a hand is defined as its percentile rank among all possible poker hands. For example, a hand that beats 75% of all poker hands has value 0.75. Each player knows only their own hand, but not their opponent's hand. Thus, Player I knows the value of x, but not the value of y, while Player II knows the value of y, but not the value of x.

The game then proceeds as follows:

- Player I can either *bet* a predetermined amount, B, or *check*.

- If Player I checks, the two hands are compared, the player with the better hand wins the pot, and the game is over. Thus, if $x > y$, then Player I has the better hand, therefore wins the pot and ends up with a net profit of $+1$. If $x < y$, then Player II wins the pot and has a net profit of $+1$, while Player I incurs a loss of -1. (The case of a tie, $x = y$, has probability 0 in this model, so it can be ignored.)
- If Player I bets, then the amount of the bet, B, is added to the pot, so that the pot now has value $B + 2$. Player II then has the option to either *call* (i.e., bet an amount B) or *fold* (i.e., give up).
  - If Player II folds, the game is over, and Player I wins the entire pot, so has a net profit of $(B + 2) - B - 1 = +1$.
  - If Player II calls, the bet amount, B, is added to the pot, the hand values are compared, the player with the better hand wins the pot, and the game is over. If $x > y$, Player I wins the pot and ends up with a profit of $2B + 2 - B - 1 = B + 1$, while Player II incurs a net loss of the same amount, i.e., a "profit" of $-B - 1$. If $x < y$, the roles are reversed, with Player II having a profit of $B + 1$, and Player I a profit of $-B - 1$.

The above situation can be represented through a *betting tree*, shown in the figure below.

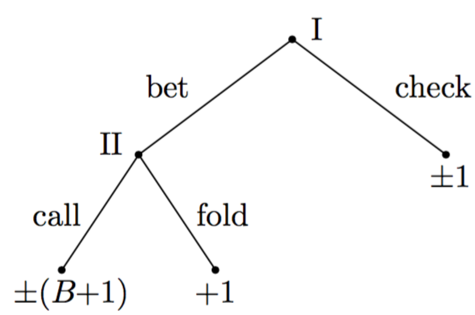

Figure 1: The Betting Tree for the von Neumann Poker Game

Optimal betting strategies. Von Neumann showed that the optimal strategies (in the sense of maximizing the expected, or long-run, profit) for the two players are as follows:

- Player I checks if $a < x < b$, and bets otherwise.
- Player II should calls if $c < y < 1$, and folds otherwise.

Here a, b, c are given as follows:

$$a = \frac{B}{(B+1)(B+4)}, \quad b = \frac{B^2 + 4B + 2}{(B+1)(B+4)}, \quad c = \frac{B(B+3)}{(B+1)(B+4)}.$$

For example, in the case $B = 2$ (i.e., the case when the bet size is equal to the size of the pot), we have $a = 1/9, b = 7/9, c = 5/9$, so the optimal strategy for Player I is to bet with a hand value in the interval $7/9 < x < 1$ or in the interval $0 < x < 1/9$, and check otherwise. The first interval corresponds to a strong hand, while the second interval represents a very weak hand. Betting with a weak hand is *bluffing*; von Neumann's result shows that bluffing is a necessary part of an optimal betting strategy, thus providing a mathematical (rather than a purely psychological) justification that bluffing "works."

Visualization. The outcomes in the von Neumann poker game can be visualized through a *payoff square* shown below. The two hand values, x and y, are represented by a point (x, y) in the unit square, and the profit (or loss) for Player I is represented by different shadings of the square. If (x, y) falls into a green-colored region, then Player I ends up with a net profit; if it falls into a red-colored region, Player I ends up with a net loss.

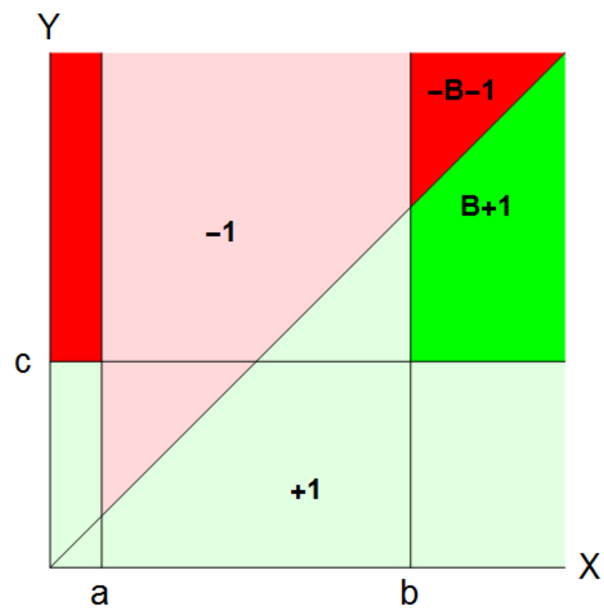

Figure 2: The Payoff Square for the von Neumann Poker Game

We have created an interactive *Mathematica* animation that allows a user to play von Neumann Poker against a computer opponent and shows the outcome of the game and the payoff square under different strategies. We plan to submit this demonstration for publication at the *Wolfram Demonstrations Project*, `http://demonstrations.wolfram.com`.

4 The Indian Poker Model

"Indian Poker" is a poker-like game in which each player sees the hands of all other players, but not his/her own hand.

To create a mathematical model of a two person Indian Poker game, we begin by following the approach of von Neumann, by letting the hands of the two players be represented by real numbers x and y, chosen independently and uniformly from the unit interval $[0, 1]$. In contrast to the von Neumann Model, here we assume that Player I knows the value of y (but not x), and thus has to base her strategy on the y-value, while Player II knows the value of x (but not y) and thus has to base his strategy on the x-value.

Optimal betting strategies. Following the approach of von Neumann, we obtain the following optimal strategy for Indian Poker. (Here B is the amount of the bet.)

- Player I checks if $a' < y < b'$, and bets otherwise.
- If player I bets, Player II calls if $x > c'$, and folds otherwise.

Here a', b', c' are given as follows:

$$a' = \frac{(B+2)^2}{(B+1)(B+4)}, \quad b' = \frac{B+2}{(B+1)(B+4)}, \quad c' = \frac{2B+4}{(B+1)(B+4)}.$$

5 Data Mining a Poker Database

The IRC Poker Database [7] is a large database of poker games that had been played online at the IRC poker channel between 1995 and 2001. The database was created by Michael Maurer and is hosted at the University of Alberta. It contains data for over 10,000,000 poker hands involving thousands of players. Among those players are several notable professional poker players, including Chris Ferguson, a *World Series of Poker* bracelet winner and co-author of the papers [5] and [6].

The figure below shows a small excerpt of this database.

Obs	Nickname	Timestamp	Player	Position	Bankroll	Action	Wonamount	Hold1	Hold2	Comm1	Comm2	Comm3	Comm4	Comm5	Preflop	Flop	Turn	River
1	KVIETYS	1001911238	2	1	52260	810	0	Ks	Jd	8h	4s	3d	Kh	5h	Brr	k	k	kc
2	ein	1001911238	2	2	9031	810	1620	7h	6d	8h	4s	3d	Kh	5h	Brc	k	k	b
3	ein	1001911368	2	1	9491	90	180	Qs	Kd	6d	As	Kh	Tc	Jh	Br	k	k	b
4	KVIETYS	1001911368	2	2	51800	90	0	Jd	Js	6d	As	Kh	Tc	Jh	Bc	k	k	c
5	DopeyTwat	1001915666	2	1	55384	10	20	7h	9h	Kc	Qs	Qh	4h	9c	Bc	k	k	k
6	offsuit	1001915666	2	2	25887	10	0	2s	Td	Kc	Qs	Qh	4h	9c	Bk	k	k	k
7	offsuit	1001915694	2	1	25892	10	0	6s	Ac	Jd	9c	8c	2s	Qs	Bc	k	k	k
8	DopeyTwat	1001915694	2	2	55379	10	20	3h	9d	Jd	9c	8c	2s	Qs	Bk	k	k	k
9	DopeyTwat	1001915759	2	1	55324	30	0	2h	Kh	4h	3h	6s	8c	9d	Bc	kc	k	k
10	offsuit	1001915759	2	2	25947	30	60	Jh	3d	4h	3h	6s	8c	9d	Bk	b	k	k
11	DopeyTwat	1001916214	2	1	56729	810	1620	Jd	9d	Kd	6d	9c	7d	Tc	Bc	kc	kr	b
12	offsuit	1001916214	2	2	26007	810	0	6s	9s	Kd	6d	9c	7d	Tc	Bk	b	bc	c
13	offsuit	1001916257	2	1	25217	90	0	Kd	8h	Ac	Qd	9h	Kh	5s	Bc	k	b	b
14	DopeyTwat	1001916257	2	2	57519	90	180	As	9s	Ac	Qd	9h	Kh	5s	Bk	k	c	c
15	DopeyTwat	1001916269	2	1	57609	10	20	7h	4h	Ac	2h	7s	8c	2c	Bc	k	k	k
16	offsuit	1001916269	2	2	25127	10	0	4s	5d	Ac	2h	7s	8c	2c	Bk	k	k	k
17	offsuit	1001916300	2	1	25122	10	20	Kh	6s	5h	Qs	8d	7h	3c	Bc	k	k	k

Figure 3: Excerpt from the IRC Poker Database

For our analysis we focused on a subset of the database containing some 200,000 hands of Texas Hold'em poker with fixed bet sizes. Within this subset, we computed hand values at each stage of the game, and we introduced a variable measuring the *aggressiveness* of a given player. We applied statistical and machine learning techniques to determine which of the various variables had the greatest effect on the profit of a player. Preliminary findings showed, as expected, a strong correlation between profit and hand value, but also, quite unexpectedly, a negative correlation between aggressiveness and net profit: "Winning" players (defined as the top 10% in terms of average profit per game) tend to be significantly less aggressive than "losing" players (defined as the bottom 10%).

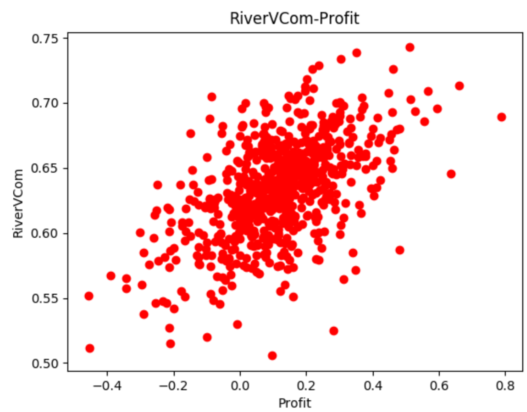

Figure 4: Correlation between profit and hand value at the River stage.

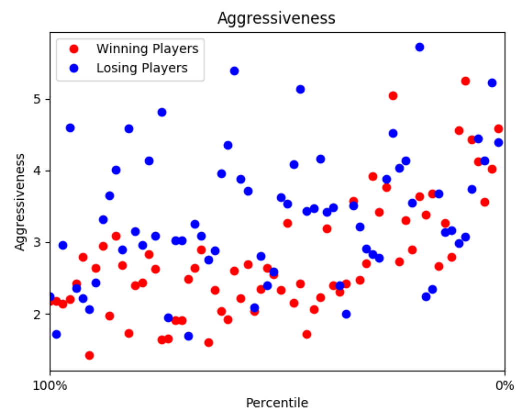

Figure 5: Aggressiveness of "winning" and "losing" players.

6 Future Directions

This project is expected to continue in the Fall 2017 semester, and possibly beyond. We will keep working towards the general goals described in the Introduction. Depending on the interests and background of the participants, we may focus more on the theoretical and game-theoretic side of the project, or the data mining part.

On the theoretical side, we plan to analyze generalizations, extensions, and variations of the von Neumann model, with the goal of determining optimal strategies.

On the data analysis side, we have begun analyzing a subset of the IRC poker database, but much remains to be explored. The database presents a one-of-a-kind real-world testing ground for all kinds of statistical analysis, and for advanced techniques such as machine-learning and artificial intelligence methods.

References

[1] Bellman, R., Blackwell, D. (1949). *Some two-person games involving bluffing.* Proc. Nat. Acad. Sci. 35, 600–605.

[2] Cassidy, J. (1998). *The last round of betting in poker.* American Math. Monthly 105, 825–831.

[3] Cassidy, J. (2015). *Early round bluffing in poker.* American Math. Monthly 122, 722–744.

[4] Cutler, W. (1975). *An optimal strategy for pot-limit poker.* American Math. Monthly 82, 367–376.

[5] Ferguson, C., Ferguson, T. (2003). *On the Borel and von Neumann poker models.* Game Theory and Applications 9, 17–32.

[6] Ferguson, C., Ferguson, T., Gawargy, C. (2007). $U(0,1)$ *two person poker models.* Game Theory and Applications 12, 17–37.

[7] Maurer, M. *IRC Poker Database.* `http://poker.cs.ualberta.ca/IRCdata/`.

[8] von Neumann, J. (1928). *Zur Theorie der Gesellschaftsspiele*, Math. Ann. 100, 295–320.

[9] von Neumann, J., Morgenstern, O. (1953). *Theory of Games and Economic Behavior.* Princeton University Press, Princeton, NJ.

[10] Newman, D.J. (1959). *A model for "real" poker.* Operations Research 7, 557–560.

[11] Zhang, H. (2010). *Two-player zero-sum poker models with one and two rounds of betting.* Penn Sci. 9, no. 1, 27–30.