Calculus, Geometry, and Probability in n Dimensions: The Geometry of Voting*

Project Report, Spring 2016

Vivek Kaushik, Aubrey Laskowski, Yukun Tan
Matthew Romney (Team Leader), A.J. Hildebrand (Faculty Mentor)†

Illinois Geometry Lab
University of Illinois at Urbana-Champaign

May 10, 2016

1 Introduction

This project is part of an ongoing program, begun in Fall 2012, aimed at seeking out and exploring interesting problems at the interface of calculus, geometry, and probability that are accessible at the calculus level, but rarely covered in standard calculus courses. Such problems are often motivated by natural questions arising in probability, statistics, and other areas.

This year (2015/2016) our focus is on the *Geometry of Voting*, an intriguing geometric approach to voting theory developed by Donald Saari during the past two decades. This approach allows one to visualize different voting methods and voting outcomes geometrically, and it helps understand and explain voting phenomena and voting paradoxes.

2 Project Goals

The general goals of this project are to:

- Learn and understand the mathematics behind the Geometry of Voting by studying some key papers and books (e.g., [3], [4]).
- Create high quality visualizations and interactive tools in Mathematica to illustrate voting phenomena and voting paradoxes using Saari's geometric approach, and make these tools widely available for educational purposes and outreach activities (e.g., through the *Wolfram Demonstrations Project*).

*An expanded version of this report is being prepared for possible publication.

†Email: ajh@illinois.edu

- Seek out and explore real-world voting examples from sports, society, and politics, and investigate the effect of different voting methods on the voting outcomes. (See [5] for one such example.)
- Investigate, theoretically and experimentally, the probabilities for various voting scenarios and voting paradoxes. (See, for example, [1] and [2].)
- Present this work to mathematical audiences, the campus community, and the broader public through conferences, open houses, and outreach events.

3 Voting Methods

Consider an election with multiple candidates, and assume each voter ranks these candidates in order of preference. A **voting method** is a method to determine an overall winner, or an overall preference ranking, given the preference rankings of the individual voters. Here are some examples of voting methods:

- **Plurality Voting:** Each voter selects one candidate. The candidate receiving the largest number of votes wins.
- **Anti-Plurality Voting:** Each voter selects one least-preferred candidate. The candidate receiving the smallest number of votes wins.
- **Borda Count:** Assign 0 points to the last-ranked candidate, 1 point to the second-last candidate, and so on. The candidate receiving the largest number of points wins.
- **Positional Methods:** This is a generalization of the above methods using an arbitrary point system.
- **Pairwise Comparison:** A candidate who beats all other candidates in a pairwise comparison is called a **Condorcet winner**, if such a candidate exists.

4 Saari Triangles

The basic idea behind Saari's geometric approach to voting theory is captured by a **Saari Triangle**, shown in Figure 1 below.

Consider an election in which each voter ranks three candidates, A, B, C, in order of preference. We represent these candidates as vertices of an equilateral triangle (the "Saari Triangle"), and we think of each voter as a point inside this triangle that represents this voter's relative preferences among these candidates. The closer a point is to a particular vertex, the more this voter prefers the candidate corresponding to this vertex over other candidates. In this way, the triangle is divided into 6 subtriangles, each corresponding to a particular preference ranking. For example, the triangle at the bottom left of Figure 1 corresponds to the ranking $A > C > B$.

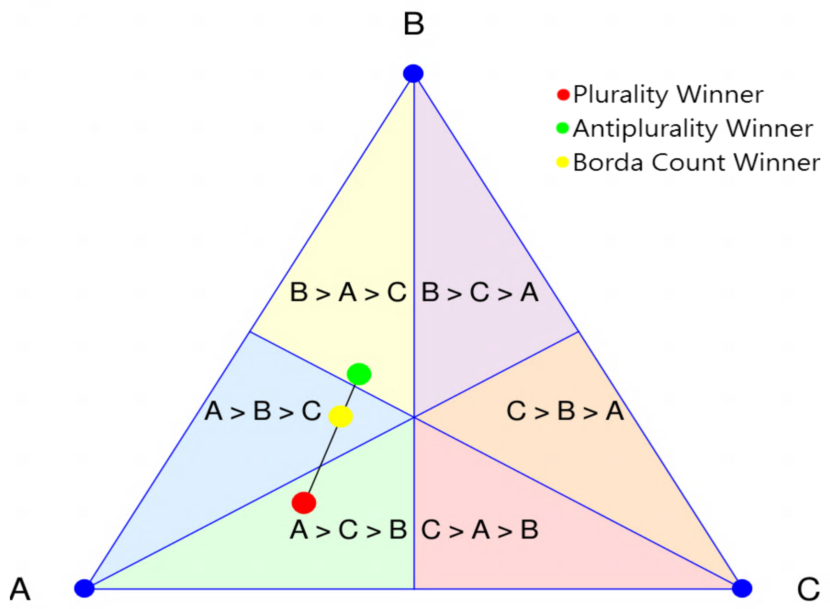

Figure 1: A Saari Triangle.

An **election outcome** can then be thought of as a point inside the triangle that represents, in an appropriate sense, an "overall" preference ranking among the entire voting population. More precisely, such a point is a convex combination of the three vertices,

$$P = \alpha A + \beta B + \gamma C, \quad \alpha + \beta + \gamma = 1,$$

where the coefficients α, β, γ represent the proportions of the votes (or points) going to the three candidates A, B, and C. For example, if $(\alpha, \beta, \gamma) = (1/2, 1/3, 1/6)$, then A is the winner of the election, with 1/2 of the vote total, B is in second place with 1/3 of the total, and C is in third place with 1/6 of the total.

A fundamental fact of voting theory is that different voting methods can yield wildly different results. In the Saari Triangle this is reflected by outcomes lying in different subtriangles. For example, in Figure 1 the red, green, and yellow points denote the outcomes under the plurality, anti-plurality, and Borda count methods, respectively, and the black line segment represents the possible voting outcomes under more general positional voting methods. Since these points fall into three different subtriangles, there are (at least) three different voting outcomes, depending on the voting method used.

The Saari Triangle app. As part of our project, we developed an interactive tool in Mathematica that illustrates the outcomes of three candidate elections using Saari triangles. The user can specify a set of voter preferences, or choose preset or randomly generated voter preferences. The tool then allows the user to choose a voting method, and it shows the election outcome under this method as a point inside the Saari triangle. The Saari Triangle app has been published at the Wolfram Demonstrations Project under the title "Three-Candidate Elections Using Saari Triangles;" see `http://demonstrations.wolfram.com/ThreeCandidateElectionsUsingSaariTriangles/`

5 Real-World Example: The 1860 U.S. Presidential Election

A fascinating case study of the effect of different voting methods on the election outcome was conducted by Tabarrok and Spector [5], using data from the 1860 U.S. Presidential Election. There were four candidates, Lincoln, Douglas, Bell, and Breckenridge. Lincoln was the clear winner of the election under the plurality method, receiving 39.78% of the popular vote, with Douglas in second place at 29.36%.

Using estimates on voter preferences among the four candidates provided by prominent historians, Tabarrok and Spector studied how the election outcome would have been affected if different voting methods had been used. They found that, surprisingly, Douglas would have been the winner of the election under most other methods. The chart in Figure 2 shows the outcomes of the election under four different voting methods.

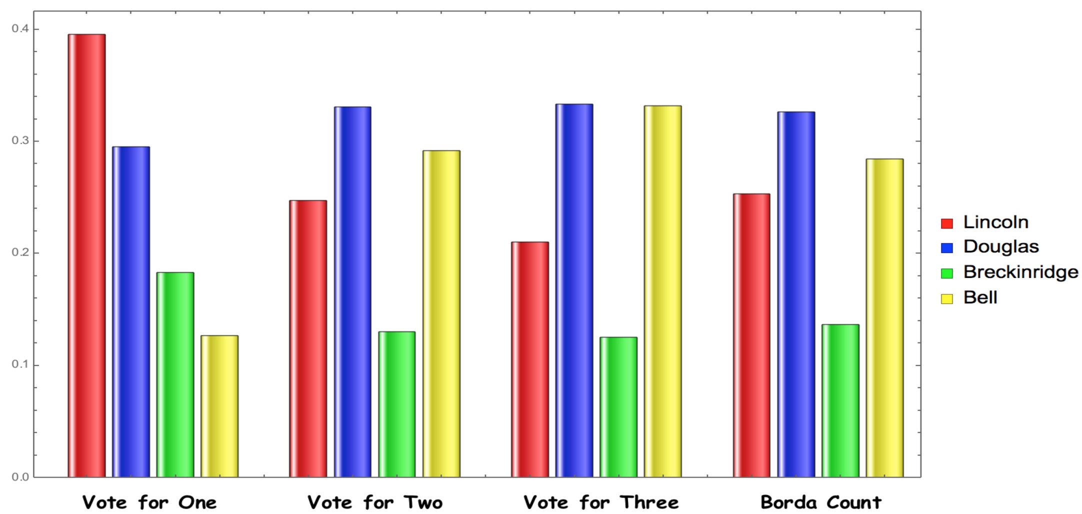

Figure 2: The 1860 U.S. Presidential Election under different voting methods.

The Lincoln Election app. As part of our project, we developed an interactive tool that illustrates the outcome of the 1860 Presidential Election under different methods. The tool allows the user to choose among several pre-set voting methods (e.g., Borda count), or choose a different voting method by explicity specifying the number of points assigned to first, second, and third place votes. It then displays the results of the election as a bar chart, or a table. The Lincoln Election app has been published at the Wolfram Demonstrations Project under the title "The 1860 US Presidential Election under Different Voting Methods;" see `http://demonstrations.wolfram.com/The1860USPresidentialElectionUnderDifferentVotingMethods`

6 Real-World Example: The 2015 AP College Football Polls

The AP College Polls provide a rich source of real-world data for the study of voting paradoxes and the effects of different voting methods. Our main focus in Spring 2016 has been on a systematic analysis of ballot data from these polls from a voting theory point of view. To our knowledge, this is the first such analysis of AP Poll data, and the largest study of paradoxes in real-world voting data that has been performed.

For our analysis, we focused on the AP College Football Polls for the 2015 season. A poll is conducted before the start of the season (pre-season poll), then weekly during the college football season, and after the end of the season (final poll), for a total of 16 such polls ("elections") in a given season.

Each poll involves around 60 voters, each of whom submits a ranked list of top 25 teams ("candidates"). The votes are tallied up using the Borda Count method (with 25 points going to the first-ranked team, 24 points to the second-ranked team, and so on), and the resulting ranked list of teams is the official AP College Football Poll ranking.

In contrast to most other college polls (such as the Coaches Poll), voter ballots in the AP Poll are publicly available. We used the site `collegepolltracker.com` to obtain ballots for the entire 2015 season. We analyzed the data for occurrences of various voting paradoxes. Our preliminary results show that there is an abundance of cycles (i.e., a set of three teams A, B, C, such that A is ranked higher than B by a majority of voters, B is ranked higher than C by a majority, and C is ranked higher than A by a majority). We also investigated other methods of aggregating votes–for example, the Borda Count method applied to only the top n teams on each voter's ballot. We found numerous instances in which the outcomes would have been different under different voting methods, including some cases in which the ranking of the top three teams was affected.

The AP Poll app. As part of our project, we developed an interactive tool that illustrates the outcomes of the 2015 AP Colllege Football Poll under alternative voting methods. The tool allows the user to assign arbitrary weights (points) to the teams ranked 1st through 25th on a given ballot and then computes the resulting overall ranking. The AP Poll app has been published at the Wolfram Demonstrations Project under the title "2015-2016 AP Football Poll Using Alternative Voting Methods;" see `http://demonstrations.wolfram.com/20152016APFootballPollUsingAlternativeVotingMethods`

References

[1] William V. Gehrlein, *Condorcet's Paradox*, Theory and Decision (1983), 15(2), 161–197.

[2] William V. Gehrlein & Peter C. Fishburn, *The Probability of the Paradox of Voting: A Computable Solution*, Jornal of Economic Theory (1976), 13, 14–25.

[3] Donald G. Saari, *Chaotic elections: A Mathematician Looks at Voting*, American Mathematical Society (2001), 33–69.

[4] Donald G. Saari & Fabrice Valognes, *Geometry, Voting, and Paradoxes*, Mathematics Magazine (1998), 71(4), 243–259.

[5] Alexander Tabarrok & Lee Spector, *Would the Borda Count Have Avoided the Civil War?*, Journal of Theoretical Politics (1999), 11(2), 261.