TITLE

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ABSTRACT

We investigate relationships between spaces of harmonic functions corresponding to the meromorphic $Q_p^{\#}$ spaces We give many analogues to the situations in the corres–ponding analytic and meromorphic spaces, and we give some examples for which the behaviors are different in the harmonic spaces.

1. Introduction

Let C denote the complex plane, let W denote the Riemann sphere, let D denote the unit disk $\{z \in \mathrm{C}\colon |z| < 1\}$, and let Σ denote the collection of all one-to-one conformal mappings of D onto itself. If f is a meromorphic function in D, we say that f is a normal function if the family $F = \{f(g(z))\colon g \in \Sigma\}$ is a normal family. We denote the family of all normal meromorphic functions by N. There is a related subfamily, the so-called "little normal functions", which is defined by

$$N_0 = \{f\colon f \text{ meromorphic in D and } \lim_{|z|\to 1}(1-|z|^2)f^{\#}(z) = 0\}$$

where $f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f.

If u is a function which is harmonic and real valued in D, we say that u is a *normal harmonic function* if the family F = $\{u(g(z)) : g \in \Sigma\}$ is a normal family. It is consequence of this definition that if u is a normal harmonic function, and if f is the analytic function f(z) = u(z) + iv(z), where v(z) is a harmonic conjugate of u(z), then f is a normal (analytic) function). However, the converse is not true, since the elliptic modular function is a normal (analytic) function for which the real part is not a normal harmonic function. We denote by N_h the family of all real harmonic normal functions.

Let w $\in$ D and let $g(z,w) = \log|\frac{1-\bar{w}z}{z-w}|$ be the Green's function in D with logarithmic singularity at w, let $u^{\#}(z) = \frac{|\mathrm{grad}\ u(z)|}{1+|u(z)|^2}$, and let dm(z) denote the Euclidean element of area in C. In it was proved that a real valued function u, harmonic in D, is a normal function if and only if

$$\sup_{z\in D}(1-|z|)u^{\#}(z) < \infty.$$

In addition to N_h , we will be considering the following classes of functions:

$UBC_h = \{u$: u real harmonic in D and
$\sup_{a\in D}\int\int_D (u^{\#}(z))^2 g(z,a)dm(z) < \infty\}$,

$UBC_{h,0} = \{u$: u real harmonic in D and
$\lim_{|a|\to 1}\int\int_D (u^{\#}(z))^2 g(z,a)dm(z) = 0\}$,

$N_{h,0} = \{u$: u real harmonic in D and
$\lim_{|z|\to 1}(1-|z|^2)u^{\#}(z) = 0\}$,

$D_h^{\#} = \{u$: u real harmonic in D and
$\int\int_D (u^{\#}(z))^2 dm(z) < \infty\}$,

and, for $0 < p < \infty$,

$Q_{h,p}^{\#} = \{u$: u real harmonic in D and
$\sup_{a\in D}\int\int_D (u^{\#}(z))^2 (g(z,a))^p dm(z) < \infty\}$,

and

$Q_{h,p,0}^{\#} = \{u$: u real harmonic in D and
$\lim_{|a|\to 1}\int\int_D (u^{\#}(z))^2 (g(z,a))^p dm(z) = 0\}$.

For $0 < p < \infty$, the spaces

$Q_p = \{f$: f analytic in D and $\sup_{a\in D}\int\int_D |f'(z)|^2 (g(z,a))^p dm(z) < \infty\}$

and the classes

$Q_p^{\#} = \{f$: f meromorphic in D and
$\sup_{a\in D}\int\int_D (f^{\#}(z))^2 (g(z,a))^p dm(z) < \infty\}$

were introduced in . It is possible to let $p = 0$ with the interpretation that

$Q_0 = \{f$: f analytic in D and $\int\int_D |f'(z)|^2 dm(z) < \infty\}$

and

$Q_0^{\#} = \{f$: f meromorphic in D and $\int\int_D (f^{\#}(z))^2 dm(z) < \infty\}$.

Under these interpretations, Q_0 is simply the usual Dirichlet space D_A and $Q_0^{\#}$ is the spherical Dirichlet space $D_A^{\#}$. We will use these interpretations in section 5.

It has been shown that the Q_p spaces and the $Q_p^{\#}$ classes have the nesting property that for $0 < p < q < \infty$ both $Q_p \subset Q_q$ and $Q_p^{\#} \subset Q_q^{\#}$). In section 4, we will give the corresponding property for the $Q_{h,p}^{\#}$ classes. In , it was proved that $D_h^{\#} \subset N_h$ (also see).

THEOREM 1. *Let u be a real harmonic function in* D, *let* $0 < r < 1$, $2 < p < \infty$, *and* $1 < q < \infty$. *The following statements are equivalent*:

(A) $u \in N_h$,

(B) $\sup_{a\in D}\frac{1}{|D(a,r)|^{1-p/2}}\int\int_{D(a,r)} (u^{\#}(z))^p dm(z) < \infty$,

(C) $\sup_{a\in D}\int\int_{D(a,r)} (u^{\#}(z))^p (1-|z|^2)^{p-2} dm(z) < \infty$,

(D) $\sup_{a\in D}\int\int_D (u^{\#}(z))^p (1-|z|^2)^{p-2}(1-|\phi_a(z)|^2)^q dm(z) < \infty$,

(E) $\sup_{a\in D}\int\int_D (u^{\#}(z))^p (1-|z|^2)^{p-2}(g(z,a))^q dm(z) < \infty$,

(F) $\sup_{a\in D}\int\int_D (u^{\#}(z))^p (log\frac{1}{|z|})^p |\phi_a'(z)|^2 dm(z) < \infty$.