For any polynomial $F(x)$, let $\bar{F}$ denote the reduction (to the least nonnegative residue) $\pmod{3}$

Question: If $F(x) = (1+x)^n$, find $\bar{F}(1)$

Write $n = \sum_{j=1}^{J} a_j e^j$

Let $A = \{j : a_j = 1\}$; $B = \{j : a_j = 2\}$

$|A| = a$; $|B| = b$.

Let $G(x) = \pi_{j=1}^{J} (1 + x^{3^j})^{a_j}$

Since $(1+x)^3 \equiv 1 + x^{3^j} \pmod{3}$, we have $G(x) \equiv F(x) \pmod{3}$.

Hence the problem reduces to finding $\bar{G}(1)$. The advantage in G is that there is no carry over, that is every monomial in the product occurs exactly once.

Now, $G(x) = \Pi_{j\in B}(1 + 2x^{3^j} + x^{2\star 3^j})\Pi_{j\in A}\left(1 + x^{3^j}\right) = \sum b_j x^j$

$$\begin{aligned} \text{Let } R &= |\{j : b_j \equiv 2 \pmod{3})\} \\ S &= \{j : b_j \equiv 1 \pmod{3})\} \end{aligned}$$

so that $\bar{G}(1) = S + 2R$.

From the expansion of $G(x)$, it is clear that if each non zero b_j, is replaced by 1, we get

$$H(x) = \Pi_{j\in B}\ (1 + x^{3^j} + x^{2\star 3^j})\ \Pi_{j\in A}\ \left(1 + x^{3^j}\right)$$

and hence $R + S = H(1) = 3^b 2^a$.

Also if we replace b_j by (-1) for $j \in R$, and b_j by $(+1)$ for $j \in S$, we end up with

$$\Pi_{j\in B}\left(1 - x^{3^j} + x^{2\star 3^j}\right)\ \Pi_{j\in A}\left(1 + x^{3^j}\right)$$

and hence $S - R = 2^a$.

Thus $S = 2^{a-1}(3^b + 1)$ and $R = 2^{a-1}(3^b - 1)$.

Hence $\bar{G}(1) = S + 2R = 2^{a-1}(3^{b+1} - 1)$