For any polynomial \( F(x) \), let \( \bar{F} \) denote the reduction (to the least nonnegative residue) \( \mod 3 \).

Question: If \( F(x) = (1 + x)^n \), find \( \bar{F}(1) \).

Write \( n = \sum_{j=1}^{J} a_j e^j \)

Let \( A = \{ j : a_j = 1 \} \); \( B = \{ j : a_j = 2 \} \)

\( |A| = a; \quad |B| = b. \)

Let \( G(x) = \prod_{j=1}^{J} (1 + x^{3^j})^{a_j} \)

Since \( (1 + x)^3 \equiv 1 + x^{3^j} \mod 3 \), we have \( G(x) \equiv F(x) \mod 3 \).

Hence the problem reduces to finding \( \bar{G}(1) \). The advantage in \( G \) is that there is no carry over, that is every monomial in the product occurs exactly once.

Now, \( G(x) = \prod_{j \in B} (1 + 2x^{3^j} + x^{2 \cdot 3^j}) \prod_{j \in A} (1 + x^{3^j}) = \sum b_j x^j \)

Let \( R = |\{ j : b_j \equiv 2 \mod 3 \}| \)

\( S = \{ j : b_j \equiv 1 \mod 3 \} \)

so that \( \bar{G}(1) = S + 2R. \)

From the expansion of \( G(x) \), it is clear that if each non zero \( b_j \), is replaced by \( 1 \), we get

\[
H(x) = \prod_{j \in B} (1 + x^{3^j} + x^{2 \cdot 3^j}) \prod_{j \in A} (1 + x^{3^j})
\]

and hence \( R + S = H(1) = 3^b 2^a. \)

Also if we replace \( b_j \) by \((-1)\) for \( j \in R \), and \( b_j \) by \((+1)\) for \( j \in S \), we end up with

\[
\prod_{j \in B} (1 - x^{3^j} + x^{2 \cdot 3^j}) \prod_{j \in A} (1 + x^{3^j})
\]

and hence \( S - R = 2^a. \)

Thus \( S = 2^{a-1}(3^b + 1) \) and \( R = 2^{a-1}(3^b - 1) \).

Hence \( \bar{G}(1) = S + 2R = 2^{a-1}(3^{b+1} - 1). \)