This problem set contains more advanced applications of the pigeonhole principle, and pigeonhole problems from math contests. Some problems require the following more general version of the pigeonhole principle:

If $kn + 1$ objects ("pigeons") are placed into $n$ boxes ("pigeonholes"), then at least one of the boxes contains more than $k$ objects.

The key in each case is to come up with an appropriate choice of the "pigeons" (e.g., the given integers) and the "pigeonholes" (e.g., congruence classes modulo 10) to which the pigeonhole can be applied.

1. **Integer sets.** For the following problems, try to classify the given integers into appropriate pairwise disjoint sets/classes, then apply pigeonhole.

   (a) Show that any set $A \subseteq \{1, 2, \ldots, 3n\}$ with at least $2n + 1$ elements contains three consecutive integers.

   **Solution:** Apply the pigeonhole principle with the triples $\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{3n - 2, 3n - 1, 3n\}$ as pigeonholes. There are $n$ such pigeonholes, and we have $2n + 1$ elements in our set $A$ to place into these pigeonholes. By the generalized pigeonhole principle, one of the pigeonholes must contain at least 3 elements. But this means that $A$ contains one of the triples $\{1, 2, 3\}, \{4, 5, 6\}, \ldots$ of consecutive integers.

   (b) Show that any set $A \subseteq \{1, 2, \ldots, 2n\}$ with at least $n + 1$ elements contains two elements that have no common prime factor.

   **Solution:** Apply pigeonhole to the pairs $\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}.$

2. **Sums/differences.**

   (a) Prove that from a set of 10 distinct two-digit integers it is possible to select two non-empty subsets whose members have the same sum.

   **Solution:** A sum of a non-empty subset of a set of ten distinct two-digit (positive) integers must be a positive integer $\leq 10 \cdot 99 = 990$, so there are at most $990$ possible values for such a sum. On the other hand, there are $2^{10} - 1 = 1023 > 990$ such subsets, so by the pigeonhole principle two of these must have the same sum.

   (b) Prove that from a set of 10 distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum. (Except for the extra requirement of disjointness, this is the same as the previous problem. Thus, you need to find a way to ensure the disjointness condition in the solution obtained by the pigeonhole principle...)
(c) (UIUC Undergraduate Math Contest 2009) Let $S$ be a set of 16 distinct positive integers, all less than 60. Show that there exist four pairwise distinct elements $a, b, c, d \in S$ such that $a + b = c + d$.

**Solution:** Let $s \in S$ be given, and consider the sets $S_1 = \{s - a : a \in A, a \leq s\}$ and $S_2 = \{a - s : a \in A, a > s\}$. Note that the elements of $S_1$ and $S_2$ are all of the form $|s - a|$ and therefore must be elements in $S$.

Now consider the cardinalities of the sets $S_1, S_2, A$. Clearly, $|S_1| + |S_2| = |A|$, so for some $i = 1, 2$ we must have $|S_i| \geq |A|/2$. Since by assumption, $|A| > (2/3)|S|$, it follows that $|S_i| > |S|/3$.

If $A$ and $S_i$ were disjoint, this would imply $|A \cup S_i| = |A| + |S_i| > (2/3)|S| + (1/3)|S| = |S|$, which is a contradiction since $A \cup S_i \subseteq S$.

Thus, $A$ and $S_i$ cannot be disjoint. Therefore there exists an element $a' \in A$ such that $a' \in S_i$. But this means that either $s - a = a'$ or $a - s = a'$. In the first case, $s$ is a sum of two elements of $A$, and in the second, $s$ is a difference of two elements on $A$. This proves the claim.

3. Digital expansions.

(a) Prove that, given any integer that is a power of 2 (for example, 1024) there exist infinitely many powers of 2 whose decimal representation ends with the digits of this integer.

**Solution:** Suppose $m = 2^h$ is a given power of 2. Then $m$ has at most $h$ decimal digits, so any integer that is congruent to $m$ modulo $10^h$ will end with the digits of $m$. Thus, it suffices to show that there exist infinitely many $n$ such that (*) $2^n \equiv 2^h \mod 10^h$.

Since there are only finitely many congruence classes modulo $10^h$, by the pigeonhole principle there exist integers $n_1, n_2$ with $h < n_1 < n_2$ such that (**) $2^{n_1} \equiv 2^{n_2} \mod 10^h$.

Dividing (**) through by $2^h$, we obtain $2^{n_1 - h} \equiv 2^{n_2 - h} \mod 5^h$, and hence $2^{n_2 - n_1} \equiv 1 \mod 5^h$. Let $p = n_2 - n_1$ and consider integers of the form $n = h + ip$. Then, for
any such \( n \) we have

\[
2^n = 2^h (2^i)^j \equiv 2^h \mod 5^h,
2^n \equiv 0 \equiv 2^h \mod 2^h,
2^n \equiv 2^h \mod 10^h.
\]

Hence \((*)\) holds for any \( n \) of the form \( n = h + ip \).

(b) Let \( x \) be an irrational real number.

(i) Prove that, given any positive integer \( k \), there exists a multiple of \( x \) whose fractional part is \( < 10^{-k} \). (The fractional part of \( x \) is defined as \( x - [x] \), where \([x]\) is the greatest integer \( \leq x \).)

**Solution:** Let \( N \) be given. Apply the pigeonhole principle with the intervals \([i/N, (i+1)/N], i = 0, 1, \ldots, N\) as “pigeonholes” and the \( N+1 \) numbers \([nx], n = 1, 2, \ldots, N+1\), as “pigeons”, to conclude that two of these numbers, say \([mx]\) and \([nx]\) with \( m < n \), fall into the same interval and hence satisfy \(|\{mx\} - \{nx\}| \leq 1/N\). But this implies that for \( h = m - n \) or \( h = n - m \) we have \([hx] < 1/N\). Since \( N \) was arbitrary, this proves the claim.

(ii) Prove that given any real numbers \( \alpha, \beta \) with \( 0 \leq \alpha < \beta \leq 1 \), there exists a multiple of \( x \) whose fractional part falls into the interval \([\alpha, \beta]\)

**Solution:** Let \( \epsilon = \beta - \alpha \), and apply the previous result to obtain a nonzero integer \( h \) such that \( d = [hx]\) satisfies \( d < \epsilon \). Since \( x \) is irrational, we cannot have \([hx] = 0\), so \( d \) must satisfy \( 0 < d < \epsilon \). It follows that the sequence \( d, 2d, 3d, \ldots \) contains at least one term in every interval of length \( \epsilon \) on the positive real axis. In particular, there exists an integer \( k \) such that \( kd \in [\alpha, \beta] \).

Now let \( n = kh \). Then, \( nx = khx = k([hx] + d) = k[hx] + kd \), and since \( k[hx] \) is an integer and \( 0 < kd < \beta \leq 1 \), it follows that \([nx] = kd \), so \([nx]\) falls in the interval \([\alpha, \beta]\), as claimed.

(c) Prove that given any finite sequence of decimal digits (e.g., 2014) there exists a power of 2 whose decimal representation **begins** with these digits. (Hint: Apply the result of the previous problem with an appropriate choice of \( x \).)

**Solution:** Let \( N \) denote the positive integer represented by the given string of digits. Adding the digit 0 at the end if necessary, we may assume \( N \) does not consist of all 9’s.

Then \( 2^n \) begins with these digits if and only if \((*)\) \( 10^kN \leq 2^n < 10^k(N+1) \) for some nonnegative integer \( k \). Taking logarithms to base 10, \((*)\) can be written as \((**)\)

\[
k + \log_{10} N \leq n \log_{10} 2 < k + \log_{10}(N+1).
\]

Our assumption that \( N \) does not consist of all 9’s ensures that the two numbers \( \log_{10} N \) and \( \log_{10}(N+1) \) have the same integer part, say \( m = \lfloor \log_{10} N \rfloor = \lfloor \log_{10}(N+1) \rfloor \), and that \( \{\log_{10} N\} < \{\log_{10}(N+1)\} \). Then, \((***)\) is equivalent to \((***)\) \( \{\log_{10} N\} < \{n \log_{10} 2\} < \{\log_{10}(N+1)\} \). By the previous problem, applied with \( x = \log_{10} 2\), \( \alpha = \lfloor \log_{10} N \rfloor + \epsilon \), and \( \beta = \{\log_{10}(N+1)\} - \epsilon \) where \( \epsilon > 0 \) is small enough to ensure that \( \alpha < \beta \). there exists a positive integer \( n \) such that \((***)\) holds. This completes the proof.

4. **Miscellaneous problems.**

(a) Show that in any group of \( n \) people there exist two that have the same number of friends.

(Answer: friendship is a mutual relationship: If A is friends with B, then B is friends with A.)

(This is a bit tricky since there seem to be just as many pigeonholes as pigeons ...)

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Solution: The possible numbers of friends are 0, 1, ..., n − 1. Since there are n people, and n possible numbers of friends, the pigeonhole principle cannot be applied right away. However, the following reasoning shows that there are, in fact, at most n − 1 possible numbers of friends: If one of the people has no friends, then nobody can have n − 1 friends (since we assumed the “friend” relation is symmetric), so the only possible numbers are 0, 1, ..., n − 2. In the other case, each of the people has at least one friend, in which case the only possible numbers are 1, 2, ..., n − 1. Thus, in each case there are at most n − 1 possible numbers of friends, and the pigeonhole principle therefore guarantees that two people must have the same number of friends.

(b) Suppose A is a collection of subsets of {1, 2, ..., n} with the property that any two sets in A have a non-empty intersection. Show that A has at most $2^n - 1$ elements.

Solution: Split the $2^n$ subsets into pairs of the form $\{A, A^c\}$. Note that a set $A$ with the given property can contain at most one element from each such pair. Since there are $2^{n-1}$ such pairs, $A$ can have at most $2^{n-1} - 1$ elements. The set $A = \{B \subseteq \{1, 2, ..., n\} : 1 \in B\}$ shows that the bound $2^{n-1}$ is best possible.

(c) (A2, Putnam 2002) Prove that, given 5 points on sphere, there exist 4 points that lie on the same hemisphere. (Hint: Consider great circles, i.e., circles that divide the sphere into two hemispheres.)

Solution: Pick 2 of the given points, and consider the circle (“great circle”) passing through these points and centered at the center of the sphere. This circle divides the sphere into two hemispheres. By the pigeonhole principle, one of these two hemispheres must contain at least 2 of the remaining 3 points, and hence a total of at least 4 points.

Challenge Problem: Decimal and Binary Digits:

Show that, for each positive integer $n$ there exists a unique positive integer whose decimal representation has exactly $n$ digits, each either 1 or 2, and whose binary representation ends in $n$ zeros.

Solution: Note first that for each $n$, there are exactly $2^n$ integers of the required form, i.e., consisting of $n$ decimal digits, each 1 or 2. To show that exactly one of these is divisible by $2^n$, it therefore suffices to show that these $2^n$ integers are all distinct modulo $2^n$.

Suppose that $a_1$ and $a_2$ are two $n$-digit integers, consisting only of 1’s and 2’s. We need to show that $a_1 - a_2$ is not divisible by $2^n$ unless $a_1 = a_2$. Clearly, if the last digits of $a_1$ and $a_2$ are different, i.e., if one of these digits is 1 and the other 2, then $a_1 - a_2$ is odd and so cannot be divisible by $2^n$. A similar argument shows that if the last $k$ digits of $a_1$ and $a_2$ are equal, but the $(k+1)$st digits are different, then $a_1 - a_2$ cannot be divisible by $2^{k+1}$. Hence, $a_1 - a_2$ can only be divisible by $2^n$ if $a_1$ and $a_2$ agree in their last $n$ digits. But since $a_1$ and $a_2$ are both $n$-digits integers, this can only happen if $a_1 = a_2$. This proves our claim.

Happy Problemsolving!