UI Putnam Training Sessions, Beginner Level
Problem Set 7: The Pigeonhole Principle  Solutions

http://www.math.illinois.edu/contests.html

The Problems  Solutions

All of these problems can be solved by an appropriate application of the pigeonhole principle. In each case, clearly identify the objects (pigeons) and the types/categories (pigeonholes) in the application of pigeonhole.

1. Parity. For the following questions try to use parity to classify the points/numbers, then apply pigeonhole.

   (a) (UIUC Freshman Math Contest 2015) Given 9 points in 3-dimensional space with integer coordinates, prove that there exist two of these points such that the midpoint of the segment joining the points also has integer coordinates.

   Solution: Classify the points according to the parity of their coordinates: (even,even,even),..., (odd,odd,odd). There are $2^3 = 8$ such “types” of points, and since we have 9 points, two of these must be of the same type, i.e., have the same parities in the $x$, $y$, and $z$ coordinates. The midpoint of the segment joining these points has integer coordinates.

   (b) (UIUC Mock Putnam Exam 2008) Let $a_1, a_2, \ldots, a_9$ be positive integers, none of which has a prime factor greater than 5. Prove that, for some $i,j$ with $i \neq j$, the product $a_ia_j$ is a perfect square.

   (Although it may look completely different, this problem is in fact closely related to the previous one! Hint: Look at the prime factorization of the numbers.)

   Solution: The given integers can only contain prime factors 2, 3, 5, so must be of the form $a_i = 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}$, with nonnegative integer exponents $\alpha_i, \beta_i, \gamma_i$.

   Now classify the given integers according to the parity of the vector of exponents $(\alpha_i, \beta_i, \gamma_i)$. There are $2^3 = 8$ possible parity vectors. Since we have 9 integers, by the pigeonhole principle, two of these integers, say $a_i$ and $a_j$, with $i \neq j$, must have the same parity vectors as exponents. But this implies that $\alpha_i + \alpha_j, \beta_i + \beta_j, \gamma_i + \gamma_j$ are all even numbers, so the number

   $$a_ia_j = 2^{\alpha_i+\alpha_j}3^{\beta_i+\beta_j}5^{\gamma_i+\gamma_j}$$

   is a perfect square.

2. Congruences. For the following questions try to use congruences to classify the points/numbers, then apply pigeonhole.

   (a) Given a set of 10 integers, show that there exist two of them whose difference is divisible by 9.
Solution: Consider congruences modulo 9. There are 9 congruence classes modulo 9, namely 0 mod 9, 1 mod 9, ..., 8 mod 9. We use these congruence classes as our “pigeonholes”, and the given numbers as the “pigeons”. Since we have 10 numbers and only 9 congruence classes, the pigeonhole principle guarantees that two of the numbers fall into the same congruence class modulo 9. The difference of these two numbers is therefore congruent to 0 modulo 9, and hence divisible by 9.

(b) Given a set of 10 integers, show that there exist two of them whose difference or sum is divisible by 16.

Solution: Similar to the previous problem, we consider congruences modulo 16, but simply taking congruence classes modulo 16 as our “pigeonholes” would not work since we have 16 such classes and only 10 integers. To overcome this issue, we take advantage of the added flexibility provided by the “difference or sum” condition. Note that the sum or difference of two numbers $h$ and $k$ divisible by 16 if and only if $h \equiv k$ or $h \equiv 16 - k$ modulo 16. Thus, instead of taking all 16 congruence classes to be our pigeonholes, we can combine each congruence class $a$ mod 16 with its “complementary” class, $16 - a$. This gives the following 9 classes modulo 16:

\[
\{0\}, \{1, 15\}, \{2, 14\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{7, 9\}, \{8\}
\]

Clearly, any integer must fall into one of these 9 congruence classes, and since we have 10 integers, by the pigeonhole principle two of these integers must fall into the same class.

3. Geometric applications. Try to divide the given region into appropriate subregions, then use these as the pigeonholes and the points as the pigeons.

(a) Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $1/2$.

Solution: Divide the triangle into four congruent equilateral triangles of side length $1/2$. Then use the pigeonhole principle to conclude that one of these must contain two points.

(b) Show that among any five points inside a $1 \times 1$ square there exist two points whose distance is at most $1/\sqrt{2}$.

Solution: Similar to the previous problem. Divide the square into four congruent subsquares of sidelength $1/2$ each.

4. Fibonacci numbers. The following problems deal with the Fibonacci sequence

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots,\]

defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 3$.

(a) Show that the sequence 01, 01, 02, 03, 05, 08, 13, 21, 34, 55, 89, 44, 33, 77, 10, 87, ... of the last two digits of the Fibonacci numbers is ultimately periodic. More generally, given any positive integer $k$, show that the sequence of numbers consisting of the last $k$ digits of the Fibonacci numbers (i.e., the sequence $F_n \mod 10^k$, $n = 1, 2, 3, \ldots$) is ultimately periodic.

Solution: Fix $k$, and let $a_n$ be the least nonnegative remainder of $F_n \mod 10^k$, so that $0 \leq a_n < 10^k$ and $a_n \equiv F_n \mod 10^k$. The Fibonacci recurrence implies
\[ a_{n+1} \equiv a_n + a_{n-1} \mod 10^k \text{ for } n \geq 2. \] This defines \( a_{n+1} \) uniquely in terms of \( a_n \) and \( a_{n-1} \). Similarly, rewriting the recurrence as \( a_{n-1} \equiv a_{n+1} - a_n \mod 10^k \) shows that \( a_{n-1} \) is defined uniquely in terms of \( a_n \) and \( a_{n+1} \). Hence, any pair \((a_n, a_{n-1})\) determines the entire sequence \(\{a_n\}\) uniquely forwards and backwards. Since there are only finitely many (namely, at most \( (10^k)^2 \)) possible values for these pairs, by the pigeonhole principle at least one these pairs must occur twice, and the sequence \((a_n, a_{n-1})\) must be periodic (i.e., form an infinite loop with no branches leading into or out of the loop).

(b) Show that, given any positive integer \( k \), there exists a Fibonacci number \( F_n \) ending in at least \( k \) zeros.

**Solution:** Define \( a_n \) as before. Since \((a_1, a_2) = (1, 1)\), there are infinitely many \( n \) with \((a_n, a_{n-1}) = (1, 1)\). For each of these \( n \), we have \( a_{n-2} \equiv a_n - a_{n-1} \equiv 0 \mod 10^k \), and hence \( F_{n-2} \equiv 0 \mod 10^k \).

(c) Consider the real number 0.1123583145943..., whose \( n \)-th digit after the decimal point is the last decimal digit of the \( n \)-th Fibonacci number. Show that this “Fibonacci constant” is a rational number.

**Solution:** By the previous part, the sequence of digits in the given number is ultimately periodic, so the number must be rational.
Challenge Problem of the Week: A Very Messy Sequence

(UI Undergraduate Math Contest 2007) Let \(a_n\) (\(n = 0, 1, \ldots\)) be a bounded sequence of positive integers that satisfies

\[
a_n \left( a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2 \right) = a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007}
\]

for all \(n \geq 2007\). Show that the sequence eventually becomes periodic.

**Solution:** Note that each term in the sequence is uniquely determined by its 2007 predecessors. The values of the sequence are bounded positive integers, there exist only finitely many choices for the 2007-tuple of values of these predecessors. Hence, by the pigeonhole principle, some such tuple must occur more than once, and at that point the sequence repeats itself.

Happy Problemsolving!