The Pigeonhole Principle (Box Principle)

- **Pigeonhole Version.** If \( n + 1 \) objects ("pigeons") are placed into \( n \) boxes ("pigeonholes"), at least one of the boxes contains more than one object.

- **Drawer Version:** If we put \( n + 1 \) objects into \( n \) drawers, at least one of the drawers must contain more than one object.

- **Categories/Types Version:** If there are \( n \) possible categories or types of objects and we have \( n + 1 \) objects, then two of these objects must be of the same type.

Examples

- Given 11 integers, there exist two that have the same last digit.
  (The “pigeonholes” are the 10 possibilities for the last digit (0, 1, \ldots, 9), and the “pigeons” are the 11 given integers. Since there are more integers than possible last digits, two of the integers must have the same last digit.)

- Given 5 pairs of integers \((x_1, y_1), \ldots, (x_5, y_5)\), there exist two that have the same parities (i.e., even or odd) in each component.
  (The “pigeonholes” are the 4 possible parity combinations \((even, even)\), \((even, odd)\), \((odd, even)\), \((odd, odd)\), and the “pigeons” are the 5 given pairs of integers. Since there are more such pairs than parity combinations, two of the pairs must have the same parity combination.)
The Problems

All of these problems can be solved by an appropriate application of the pigeonhole principle. In each case, clearly identify the objects (pigeons) and the types/categories (pigeonholes) in the application of pigeonhole.

1. **Parity.** For the following questions try to use parity to classify the points/numbers, then apply pigeonhole.

   (a) (UIUC Freshman Math Contest 2015) Given 9 points in 3-dimensional space with integer coordinates, prove that there exist two of these points such that the midpoint of the segment joining the points also has integer coordinates.

   (b) (UIUC Mock Putnam Exam 2008) Let $a_1, a_2, \ldots, a_9$ be positive integers, none of which has a prime factor greater than 5. Prove that, for some $i, j$ with $i \neq j$, the product $a_i a_j$ is a perfect square.

   (Although it may look completely different, this problem is in fact closely related to the previous one! Hint: Look at the prime factorization of the numbers.)
2. **Congruences.** For the following questions try to use congruences to classify the points/numbers, then apply pigeonhole.

(a) Given a set of 10 integers, show that there exist two of them whose difference is divisible by 9.
(b) Given a set of 10 integers, show that there exist two of them whose *difference or sum* is divisible by 16.
3. **Geometric applications.** Try to divide the given region into appropriate subregions, then use these as the pigeonholes and the points as the pigeons.

(a) Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $1/2$.

(b) Show that among any five points inside a $1 \times 1$ square there exist two points whose distance is at most $1/\sqrt{2}$. 
4. **Fibonacci numbers.** The following problems deal with the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots,$$

defined by \(F_1 = F_2 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for \(n \geq 3\).

(a) Show that the sequence 01, 01, 02, 03, 05, 08, 13, 21, 34, 55, 89, 44, 33, 77, 10, 87, \ldots of the last two digits of the Fibonacci numbers is ultimately periodic. More generally, given any positive integer \(k\), show that the sequence of numbers consisting of the last \(k\) digits of the Fibonacci numbers (i.e., the sequence \(F_n \mod 10^k, n = 1, 2, 3, \ldots\)) is ultimately periodic.

(b) Show that, given any positive integer \(k\), there exists a Fibonacci number \(F_n\) ending in at least \(k\) zeros.

(c) Consider the real number 0.1123583145943\(\ldots\), whose \(n\)-th digit after the decimal point is the last decimal digit of the \(n\)-th Fibonacci number. Show that this “Fibonacci constant” is a rational number.
Challenge Problem of the Week: A Very Messy Sequence

(UI Undergraduate Math Contest 2007) Let \( a_n \) \((n = 0, 1, \ldots)\) be a bounded sequence of positive integers that satisfies

\[
a_n \left( a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2 \right) = a_{n-1}^3 + a_{n-2}^3 + \cdots + a_{n-2007}^3
\]

for all \( n \geq 2007 \). Show that the sequence eventually becomes periodic.

Happy Problemsolving!