Problems involving parity and invariants, II Solutions

Most of the following problems are from the book “The Art of Problem-solving” by Paul Zeitz.

1. Parity problems. Try to solve the following problems using with parity arguments.

(a) Show that if every room in a house has an even number of doors, then the number of outside doors must be even as well.

Solution: (Zeitz, 3.4.30) Number the rooms consecutively as 1, 2, . . . , n and let $a_i$ denote the number of doors in room $i$. Consider the sum $S = \sum_{i=1}^{n} a_i$. Each interior door is counted exactly twice in $S$, while each exterior door is counted once. Hence $S$ is even if and only if the number of exterior doors is even. Since by assumption, each $a_i$ is even, $S$, and hence the number of exterior doors, must be even.

(b) How many $n \times n$ matrices with entries 0 and 1 are there such that each row contains an odd number of 1s?

Solution: Answer: $2^{n(n-1)}$. Place 0’s and 1’s arbitrarily in the first $n-1$ columns rows; there are $2^{(n-1)n}$ ways to do that. The parity condition then determines the entries in the $n$-th column.

(c) 127 people play in a tennis tournament. Prove that, at the end of the tournament, the number of people that have played an odd number of games is even.

Solution: (Zeitz, Example 3.4.7) Let $g_1, g_2, \ldots, g_{127}$ denote the number of games played by each of the 127 people. Consider the sum $g_1 + \cdots + g_{127}$. Since each game is counted exactly twice (namely, in each of the two players’ counts), this sum is twice the total number of games, and hence even. This is only possible if the number of odd terms among $g_1, g_2, \ldots, g_{127}$ is even.

2. Problems involving invariants. For these problems, try to find an appropriate invariant.

(a) Consider the following game played on a list of three numbers. At each move you select two of the three numbers on the list, say a and b, and replace them by $0.6a + 0.8b$ and $0.8a - 0.6b$. Can the numbers (3, 4, 5), can be transformed to (4, 6, 12) after a finite number of such moves?

Solution: (Zeitz, 3.4.26) No. Interpret the tuples as vectors in 3-dimensional space, and note the norms of $(a, b, c)$ and $(0.6a + 0.8b, 0.8a - 0.6b, c)$ are the same. Thus the norm is an invariant, and since $(4, 6, 12)$ does not have the same norm as $(3, 4, 5)$, this point cannot be reached.

(b) Consider a game played on pairs of integers, where at each stage you can transform a pair $(x, y)$ into one of the pairs $(x - y, y)$, $(x + y, y)$, and $(y, x)$. Starting with the pair $(1, 4)$, is it possible to get the pair $(2010, 2013)$ after finitely many moves?
Solution: No. Consider the greatest common divisor (gcd). Since the gcd of \((x \pm y, y)\) is the same as that of \((x, y)\), the gcd is an invariant in this game. Since \((1, 4)\) has gcd 1 while \((2010, 2013)\) has gcd 3, the latter pair cannot be reached.


(a) Consider a matrix consisting of infinitely many rows and \(n\) columns defined as follows. The top row consists of an arbitrary finite sequence of integers, not necessarily distinct. Given a row with entries \(a_1, a_2, \ldots, a_n\), the \(i\)-th entry in the following row is defined as the number of occurrences of the number \(a_i\) among the entries \(a_1, a_2, \ldots, a_n\). For example, if the given row has entries 1, 2, 1, 3, the following row has entries 2, 1, 2, 1. Prove that, from some point onwards, all rows must be identical.

Solution: The proof follows from the following observations:

1. First, from the second row onwards, any entry \(k\) in a given row of that matrix must appear at least \(k\) times in that row. This is because, by the construction of the matrix, \(k\) counts the number of occurrences of a given entry in the previous row, and for every occurrence of that entry in the previous row the corresponding entry in the given row is \(k\).

2. As a consequence of (1), from the second row onwards, given any entry, say \(k\), the entry in the same column in the next row must be at least equal to \(k\). Hence, in any given column the entries in that column, from the second term onwards, form a non-decreasing sequence.

3. If \(n\) is the number of columns in the matrix (which, by hypothesis, is finite), then all entries in the matrix from the second column onwards are bounded from above by \(n\).

4. From (2) and (3), we see that the entries in each column from the second row onwards form a non-decreasing and bounded sequence, and therefore converge to a limit. Since the entries are all integers, this means that the terms of each column sequence must be equal from some point onwards. Since there are finitely many columns, this implies that from some point onwards, all column sequences are stationary, and therefore that the rows beyond that point are identical.

(b) Call sequence of \(2n + 1\) integers (not necessarily distinct) balanced if it has the property that if we remove any one of these \(2n + 1\) numbers, the remaining \(2n\) numbers can be divided into two groups of \(n\) numbers each, having the same sum. Obviously, if the integers in the sequence are all equal, then this property holds. Show that such sequences are the only balanced sequences of \(2n+1\) integers; i.e., any balanced sequence \(a_1, a_2, \ldots, a_{2n+1}\) must satisfy \(a_1 = a_2 = \cdots = a_{2n+1}\).

Solution: Clearly, for any integer \(a\), the numbers \(a_1 - a, a_2 - a, \ldots, a_{2n+1} - a\) also satisfy the conditions of the problem. We may therefore assume that one of the given numbers is zero, and we need to show that all of them are zero.

Let \(S\) denote the sum of all \(2n + 1\) numbers \(a_i\). The hypothesis implies that, for each \(i\), \(S - a_i\) is twice an integer, and hence must be even. This is only possible if all \(a_i\) have the same parity. Since, by our assumption, one of the numbers \(a_i\) is zero, all numbers \(a_i\) must be even. Dividing each \(a_i\) by 2, we obtain a new set of numbers \(a'_i = a_i/2, i = 1, \ldots, 2n + 1\), which again satisfies the conditions of the problem, with at least one of the numbers being zero. We conclude as before that the numbers \(a'_i\) must all be even, i.e., that 2 divides \(a'_i\) for all \(i\), or equivalently, that \(2^2\) divides \(a_i\) for all \(i\). Continuing this process, we see that, for any positive integer \(k\), \(2^k\) divides all \(a_i\). But this is only possible if all \(a_i\) are zero.
**Challenge Problem of the Week**

**Parity of binomial coefficients.** We know that the binomial coefficients \( \binom{n}{k} \) are all integers. Can we say something about their parity **without actually calculating the coefficients**? This is the goal of this week’s challenge:

*Of the 2018 binomial coefficients \( \binom{2017}{k}, k = 0, 1, \ldots, 2017, \) how many are odd? (Hint: Determine first the number of odd binomial coefficients \( \binom{n}{k} \) in the case when \( n \) is a power of 2.)*

**Solution:** We will prove a more general result:

**Theorem.** The number of odd binomial coefficients \( \binom{n}{k}, k = 0, 1, \ldots, n, \) is equal to \( 2^s \), where \( s \) is the number of 1’s in the binary expansion of \( n \), i.e., the number of distinct powers of 2 needed to represent \( n \).

To solve the challenge problem, we represent 2017 in binary, i.e., as a sum of powers of 2:

\[
2017 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^0
\]

This representation involves 7 powers of 2, so by the theorem it follows that 27 = 128 coefficients \( \binom{2017}{k} \) are odd.

**Proof of Theorem.** We use the fact that the binomial coefficients \( \binom{n}{k}, k = 0, 1, \ldots, n, \) are the coefficients of the polynomial \((1 + x)^n\). Thus we need to determine how many of these coefficients are odd. To this end we use polynomial congruences, which are termwise congruences for the coefficients of the polynomials. For example, we have the polynomial congruence

\[
1 + 2x + 3x^2 + 4x^3 + 5x^4 \equiv 1 + x^2 + x^4 \mod 2
\]

since 2 \equiv 0, 3 \equiv 1, 4 \equiv 0, and 5 \equiv 1 modulo 2. The key observation is that replacing a polynomial by one that is congruent to it modulo 2 (in the above polynomial congruence sense) does not change its number of odd coefficients.

As a first step we show by induction that, for each nonnegative integer \( k \),

\[
(1 + x)^{2k} \equiv 1 + x^{2k} \mod 2.
\]

In the base case \( k = 0 \), (1) holds trivially. Now let \( k \geq 0 \) and assume (1) holds for \( k \). Then

\[
(1 + x)^{2k+1} = \left( (1 + x)^{2k} \right)^2 \equiv \left( 1 + x^{2k} \right)^2 \quad \text{(by (1))}
= 1 + 2x^{2k} + (x^{2k})^2 \equiv 1 + x^{2k+1},
\]

which proves (1) for \( k + 1 \) and completes the induction.

Since the polynomial on the right of (1) has exactly two odd coefficients, by the above observation this shows that the same is true for the polynomial on the left, \((1 + x)^{2k}\). Thus, we have shown that when \( n \) is a power of 2, exactly 2 of the binomial coefficients \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \), are odd. This proves the theorem in the special case \( n = 2^k \).

We now consider the general case. Given a positive integer \( n \), write \( n \) as (⋆) \( n = 2^{k_0} + \ldots + 2^{k_s} \), where \( 0 \leq k_0 < k_1 < \cdots < k_s \). (This is just the binary expansion of \( n \) in disguise.) We need to show that
exactly $2^s$ of the terms in the expansion of $(1 + x)^n$ are odd. We have

\begin{equation}
(1 + x)^n = (1 + x)^{2^{k_0}} (1 + x)^{2^{k_1}} \ldots (1 + x)^{2^{k_s}} \\
\equiv \left(1 + x^{2^{k_0}}\right) \left(1 + x^{2^{k_1}}\right) \ldots \left(1 + x^{2^{k_s}}\right) \mod 2 \quad \text{(by (1))}
\end{equation}

\[ = \sum_{m \in S} x^m, \]

where each of the exponents $m$ occurring in the last step is a sum of a subset of the powers $2^{k_0}, \ldots, 2^{k_s}$. Since the numbers $k_i$ are distinct, all of these subsets have distinct sums (by the uniqueness of a binary expansion). Since there are $2^s$ such subsets, the polynomial on the right of (2) consists of $2^s$ distinct terms $x^m$, each with coefficient 1. It follows that this polynomial, and hence the polynomial $(1 + x)^n$, has exactly $2^s$ odd terms. This completes the proof.