About this Problem Set

The problems in this set illustrate a simple, but very powerful, problem solving technique, namely the use of “parity” and “invariants”. Parity refers to the classification of integers as “even” or “odd”. For most problems, all you need to know is what an even or odd integer is, and what happens to the parity when you add or multiply integers. If you want, you can think of parity as a very special case of a congruence (even numbers are those congruent to 0 mod 2, odd numbers are those congruent to 1 mod 2), but this would be overkill for most of the problems here.

- **Parity**: Parity is the even/odd character of an integer. Here are some basic properties of parity.

  - **Formal definition of “even” and “odd”**: An integer is even if and only if it is of the form \( n = 2k \), where \( k \) is an integer; it is odd if and only if it is of the form \( n = 2k + 1 \), where \( k \) is an integer. Any integer is either even or odd, but not both.
  
  - **Changing + into − or vice versa does not affect the parity.** Another “obvious” fact that is surprisingly useful. For example, an expression of the form \( a_1 \pm a_2 \pm \cdots \pm a_n \), with integers \( a_i \) and arbitrary ± signs will have the same parity \( a_1 + a_2 + \cdots + a_n \).

  - **A sum of integers is odd if and only if there are an odd number of odd terms.** Yet another “obvious” fact with lots of applications.

  - **A product of integers is odd if and only if all factors are odd.**

- **Invariants**: A technique typically used for problems involving a repeated process. The idea is to find an “invariant”, i.e., some quantity or property (e.g., parity) that does not change when performing each step in the process. Thus, at the end of the entire process this quantity is the same as it was at the beginning, and the latter can usually be easily computed. One of the most common invariants is parity, so if you see a problem involving a repeated process look for quantities whose parity does not change.

Practicing the Write-up of Solutions

In contrast to high school contest like AMC, where only the answers matter, in college level math contests properly written up solutions are expected. **You won’t get any credit for an answer alone, and you may lose a significant amount of credit if your write-up has logical gaps, is poorly presented, or illegible.** Developing strong proof-writing skills takes a lot of practice, but will pay off in the long run. Such skills are useful not only for math contests, but also in many upper level math courses, e.g., Math 347, 416, 447.

**Use these problems to practice the proper write-up of solutions.** The problems all require a properly written up proof, but at the same time most of these proofs are simple enough in structure to require no more than a couple of sentences. Thus, they provide an ideal opportunity to start practicing the write-up of proofs; take advantage of this opportunity! **We will be happy to look at your write-up, and provide feedback during the training session.** Alternatively, if you turn in your work at the end of the session, we can look at it in detail until the next session.
The Problems

1. **Warmup problems.** All of these can be done with simple parity (even/odd) arguments. (Alternatively, you can also use congruences modulo 2.)

   (a) Suppose $a_1, \ldots, a_{10}$ are each $+1$ or $-1$. If the product of all $a_i$'s is equal to 1, can the sum be equal to 0?

   (b) Show that the equation $x^2 + 2013x + 2014y = 1$ has no integer solutions. (Hint: The right side, 1, is an odd number. What can you say about the left side?)

   (c) Consider the sequence $1\ 2\ 3\ 4\ \ldots\ 2013\ 2014$. Is it possible to insert $+$ or $-$ signs in the spaces between any two consecutive numbers so that the resulting expression is equal to 0? For example, with the sequence $1\ 2\ 3\ 4$ this is possible: Inserting $-, -, +$ gives $1 - 2 - 3 + 4 = 0$. Can the same be done with the given sequence consisting of the first 2014 positive integers?
2. Some applications: Rankings, tournaments, etc: Again, using parity does the trick!

(a) Up/down moves in a Top 25 poll. Suppose 25 football teams are ranked each week from first (#1) to last (#25), with no ties allowed. Naturally, these rankings change from week to week. For example, a team ranked #5 may move to #2 (a move up by 3 spots), or a team ranked #4 may move to #11 (a move down by 7 spots).

Is it possible for each team to always move up or down an odd number of spots from one week to the next?

(b) Scheduling games for the “Big Thirteen” Conference. Consider a “Big Thirteen” football conference with exactly 13 teams.

Is it possible to schedule games so that each team plays exactly 9 games within the conference?

(Note that we impose no restrictions on how these games are scheduled: For example, a team could play all of its 9 games against the same opponent, or it could play all of its games against different opponents, or some combination of the two.)

(c) Scheduling a tennis tournament. 127 people play in a tennis tournament.

Prove that, no matter how the tournament is played, the number of people that have played an odd number of games is always even.

(As before, there are no restrictions on how the tournament games are scheduled. For example, it could be a round-robin style tournament, where every player plays one game against every other player, or it could be a knockout tournament, where a player is eliminated as soon as he/she loses a match, and anything in between.)
3. **Number games and invariants.** The following problems involve games on lists of numbers (or lattice points) that terminate after finitely many steps. Try to solve these by finding an appropriate invariant.

(a) Suppose you have a list of 5 1’s and 5 0’s, written down next to each other.
You play the following game: At every stage, you pick two numbers from the current list, erase
these two numbers and replace them by a 0 if the two numbers are the same, and by a 1 if they
are different. Thus, each stage reduces the number of numbers on the list by 1, and after 9 such
moves there is only one number left. *Prove that this surviving number is 1.*

(b) Suppose the numbers 1, 2, 3, ..., 2013 are written down on a sheet of paper and the following game
is played. At each move, two of the numbers are crossed out and replaced by the absolute value of
their difference. After 2012 such moves only one number is left. *Prove that this remaining number
must be odd.*

(c) Consider a “walk” on the lattice points (i.e., points with integer coordinates) in the plane, with
moves given by the vectors (1, 1), (1, −1), (−1, 1), (−1, −1). *If you start at the point (0, 0), is it possible to reach the point (2014, 2015) with finitely many moves of this type?*
Fun Problem of the Week

The Light Switch Problem. Consider an infinite row of light switches, labeled 1, 2, 3, 4, \ldots.

- At the beginning, all switches are turned off.
- A first person comes along and flips all switches (so that now all switches are in the “on” position).
- A second person comes along and flips every second switch (so that now switches 2, 4, 6, 8, \ldots are in the “off” position, while the other switches are still “on”).
- A third person comes along and flips every third switch (thus, turning switch 3 from “on” to “off”, switch 6 from “off” to “on”, switch 9 from “on” to “off”, etc.).
- The process continues in this manner, with the \( n \)-th person flipping every \( n \)-th switch.

Which of the switches eventually remain in the “on” position? (For example, switch 1 remains “on” after the first step of the process since this switch is only touched by the first person.)

Happy Problemsolving!