About this Problem Set

- **Prerequisites:** (ALMOST) NONE! These problems require no specialized knowledge in number theory except for the use of “congruence magic” or “modular arithmetic”, a simple but very powerful tool explained below.

- **DO NOT USE CALCULATORS!** All problems are intended to be done without a calculator. You won’t be able to use calculators in any of our math contests, so try to get into the habit of doing things without a calculator. By the same token, don’t try to approach the problems by brute force hand calculations. The numbers are large enough to make a brute force approach impractical, whereas with the “right” technique (using congruences—see below), the calculations required are minimal.

Modular Arithmetic Basics

Here is everything you need to know about modular arithmetic and “congruence magic”:

- **Congruences:** We say $a$ is congruent to $b$ modulo $m$, and write $a \equiv b \mod m$, if $a$ and $b$ have the same remainder when divided by $m$, or equivalently if $a - b$ is divisible by $m$. Equivalently, the congruence notation $a \equiv b \mod m$ can be thought of as a shorthand notation for the statement “there exists an integer $k$ such that $a = b + km$.”

Here are some examples to illustrate this notation:

1. $5 \equiv 17 \mod 3$
   (since 5 and 17 both have remainder 2 when divided by 3, or equivalently, since $17 - 5 = 12$ is divisible by 3).

2. $10 \equiv -4 \mod 7$
   (since $10 - (-4) = 14$ is divisible by 7)

3. $2013 \equiv 0 \mod 3$
   (since $2013 - 0 = 2013$ is divisible by 3)

4. $n \equiv 0 \mod 2$ if $n$ is even, and $n \equiv 1 \mod 2$ if $n$ is odd.

5. $n \equiv d \mod 10$ if $n$ has $d$ as last decimal digit
   (since then $n = d + 10k$ for some integer $k$, and hence $n \equiv d \mod 10$)

- **Modular arithmetic:** The key fact about congruences is that congruences to the same modulus can be added, multiplied, and taken to a fixed positive integral power. For example, since $6 \equiv -1 \mod 7$, we have $6^{1000} \equiv (-1)^{1000} = 1 \mod 7$. This type of manipulation is called modular arithmetic or congruence magic, and it allows one to quickly calculate remainders and last digits of numbers with thousands of digits.
• **Congruence magic: An example.** Consider the problem of finding the last digit of $2013^{2014}$ (which is a number with thousands of decimal digits). The last decimal digit is the same as the remainder modulo 10, so the problem can be restated as finding $2013^{2014} \mod 10$, i.e., the remainder of $2013^{2014}$ modulo 10. Here is how to do this with minimal computations using congruence magic:

$$
2013 \equiv 3 \mod 10 \quad \text{(reduce base mod 10)}, \\
2013^{2014} \equiv 3^{2014} \mod 10, \quad \text{(simplify the problem)}, \\
3^4 = 81 \equiv 1 \mod 10, \quad \text{(find small power that is 1 mod 10)},
$$

$$
3^{4k} \equiv 1^k = 1 \mod 10 \quad \text{(for any pos. integer } k) \\
3^{2012} = 3^{4 \cdot 503} \equiv 1 \mod 10 \quad \text{(take } k = 503 \text{ to get an exponent close to 2014)}
$$

$$
3^{2014} = 3^{2012} \cdot 3^2 \equiv 1 \cdot 9 = 9 \mod 10
$$

Thus the last digit of $2013^{2014}$ is 9.

• **Fermat’s Theorem.** An important special congruence is given by Fermat’s Theorem, which says that if $p$ is a prime number and $a$ is not divisible by $p$, then $a^{p-1} \equiv 1 \mod p$.

For example, since 2017 is a prime number, we have $a^{2016} \equiv 1 \mod 2017$ for any integer $a$ that is not a multiple of 2017.

**Important note:** Fermat’s theorem only applies if the modulus, $p$, is a prime number. In particular, the theorem cannot be applied in the above example (which required computing $2013^{2014} \mod 10$) since the modulus, 10, is not a prime number. (In fact, from the above calculation, we have $3^9 \equiv 3 \cdot 3^8 \equiv 3 \cdot 1 \equiv 3 \mod 10$, whereas Fermat’s theorem, if it were applicable, would give $3^9 \equiv 1 \mod 10$.)
The Problems

1. Warmup: Practice with congruence notations:

   (a) Which of the following congruences are true?

      (i) $37 \equiv -13 \mod 3$.

      (ii) $107 \equiv 72 \mod 7$.

      (iii) $n \equiv -n \mod 2$ for any integer $n$.

      (v) $n \equiv 2n \mod 2$ for any integer $n$.

      (vi) $n(n + 1) \equiv 0 \mod 2$ for any integer $n$.

   (b) What is the remainder of $100 \cdot 101 \cdot 102 \cdot 103$ when divided by 99? (This requires almost no computations if approached the “right” way! Don’t try to multiply out the numbers.)

   (c) What is the remainder of $100 \cdot 99 \cdot 98 \cdot 97$ when divided by 101? (Again, don’t try to multiply out the numbers.)
2. **Congruence magic, I: Quick computation of remainders and last digits:** Use congruence magic to compute the following quickly and painlessly, *without multiplying large numbers*.

(a) The remainder of $2013^{2014}$ when divided by 2014?

(b) The remainder of $1001^{1001}$ when divided by 3.

(c) The last decimal digit of $3^{347}$.

(d) The last *two* decimal digits of $99^{1001}$.

(e) (UI Freshman Math Contest, 2012) Prove that there exists no power of 2 whose decimal represenation ends in the digits 2012. (Hint: Consider congruences modulo 8.)
3. Congruence magic, II: Divisibility tests: Congruences can be used to justify, and generalize, the familiar divisibility tests by 3 and 9. This exercise guides you through the process.

(a) Given an integer \(n\) with decimal expansion \(n = (a_k a_{k-1} \ldots a_0)_{10}\), express \(n\) as a sum involving powers of 10.

(b) Now take this expression, reduce modulo 9, and simplify as much as possible. Show that the result is the sum of digits, \(a_0 + a_1 + \cdots + a_k\).

In other words, if \(s(n)\) denotes the sum of decimal digits of \(n\), then we have \(s(n) \equiv n \mod 9\). In particular, \(n\) is divisible by 9 (i.e., \(n \equiv 0 \mod 9\)) if and only if \(s(n)\) is divisible by 9 (i.e., \(s(n) \equiv 0 \mod 9\)). This is the divisibility test for 9.

(c) An application: Show that any integer that contains each of the nine digits 1, 2, \ldots, 9 exactly once (for example 359261784) is divisible by 9.

(d) Another application. Let \(n = 4 \ldots \ldots 4\) be a number consisting of 4444 4’s when written in decimal. What is the remainder of \(n\) when divided by 9?

(e) Divisibility test for 11. Using a similar approach as above, find a divisibility test for 11. Then use this test to find the remainder of 123456789 when divided by 11.
Fun Problem of the Week

| The Four 4’s Problem (International Mathematical Olympiad, 1975). Let A = 44444444. Let B be the sum of the decimal digits of A. Let C be the sum of the decimal digits of B. Let D be the sum of the decimal digits of C. What is D? |

Happy Problemsolving!