The Problems

1. **Warmup 1. Counting icecream combinations at Cocomero.** Cocomero has 16 varieties of frozen yogurt and 36 different toppings. How many combinations of frozen yogurt and toppings can you create using:

   (a) A single frozen yogurt flavor and one topping.
   **Solution:** \(16 \cdot 36\)

   (b) A single frozen yogurt flavor and two different toppings.
   **Solution:** \(16 \cdot \binom{36}{2}\)

   (c) A single frozen yogurt flavor and any number of toppings (including no toppings at all).
   There are two ways to approach this problem. The first approach leads to a sum of binomial coefficients. The second approach leads to a very simple expression involving just one term. Try to find both approaches!
   **Solution:** \(16 \cdot 2^{36} - 2^{40}\)

   (d) The real thing: At least one yogurt flavor and any number of toppings.
   As before, there are two approaches, one leading to a messy sum, while the other “slick” approach leads to a much simpler expression. Try to find the slick method.
   **Solution:** \((2^{16} - 1) \cdot 2^{36}\)

2. **Warmup 2. Counting pizza combinations at MiaZa’s.** MiaZa’s pizza menu offers 7 sauces, 15 premium toppings, and 25 free toppings. How many different pizzas can you create using:

   (a) One sauce, one premium topping, and one free topping.
   **Solution:** \(7 \cdot 15 \cdot 25\)

   (b) One sauce, one premium topping, and two free toppings.
   **Solution:** \(7 \cdot 15 \cdot \binom{25}{2}\)

   (c) One sauce, one premium topping, and up to four free toppings (the maximum number you can choose without extra charge).
   **Solution:** \(7 \cdot 15 \cdot (1 + \binom{25}{1} + \binom{25}{2} + \binom{25}{3} + \binom{25}{4})\)

   (d) BONUS QUESTION: Which is the bigger number: The number of frozen yogurt combinations at Cocomero or the number of pizzas you can create at MiaZa’s without extra charge? Try to answer this without using a calculator!
   **Solution:** The first number is (**) \((2^{16} - 1)2^{36} > 2^{15} \cdot 2^{36} = 2^{51}\). The second number is (***) \(9 \cdot 7 \cdot 15 \cdot 2^{25}\), since the number of choices of free toppings with the “up to four” restriction is less than the number of such choices without such a restriction, i.e., \(2^{25}\). Now, \(9 \cdot 7 \cdot 15 \cdot 2^{25} < 2^4 \cdot 2^3 \cdot 2^4 \cdot 2^{25} = 2^{4+3+4+25} = 2^{36}\), which is (a lot) smaller than (**).
3. Counting committees: Consider a group of 30 students.

(a) How many ways are there to form a 3-person government consisting of a president, a vice-
    president, and a treasurer?

Solution: \(30 \cdot 29 \cdot 28\)

(b) How many ways are there to form a 3-person committee?

Solution: \(\binom{30}{3}\)

(c) How many ways are there to form a non-empty committee with an arbitrary nonzero number
    of members?

Solution: \(2^{30} - 1\)

(d) How many ways are there to form a 10-person committee and elect a chair of this committee?
    Try to come up with the answer in two ways: (i) by first picking a chair from the group of 30, then selecting
    the remaining 9 members from the remaining 29 students in the group; (ii) by first picking a 10-person committee, and then picking a chair from this committee. Show that the answers you get by these two methods are the same.

Solution: Method (i) gives \(30 \cdot \binom{29}{9}\) as answer; method (ii) gives \(\binom{30}{10} \cdot 10\). Since \(\binom{30}{10} = 30 \cdot 29 \cdot 28 \cdots 21/10! = (30/10)\binom{29}{9}\), the two answers are the same.

4. Counting words, I. For the following problems, any finite string of letters counts as a “word”.
   Unless otherwise indicated, all words are assumed to be in upper case (i.e., capital letters). For
   example, with this interpretation there are exactly six 3-letter words that contain each of the letters
   HAL: HAL, HLA, AHL, ALA, LHA, LAH. Assume there are 26 letters in the alphabet.

(a) How many words can be obtained by rearranging the 6 letters of the word “PUTNAM”? (Include
    the word PUTNAM itself in the count.)

Solution: \(6!\)

(b) How many 4-letter words can be obtained using the letters of the word “PUTNAM”, with each letter used
    at most once?

Solution: \(6 \cdot 5 \cdot 4 \cdot 3\)

(c) How many 4-letter words can be obtained using the letters of the word “PUTNAM”, if repetition of letters is allowed? (For example, “NANA” and “PPPP” are words included in this count.)

Solution: \(6^4\)

(d) How many 4-letter words can be obtained using the letters of the word “PUTNAM”, if repetition of letters is allowed, but adjacent letters cannot be the same? (Here, “NANA” is counted, but “NAAN” is not since it contains two adjacent letters A.)

Solution: \(6 \cdot 5^3\)

5. Counting words, II.

(a) How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition of letters is allowed?

Solution: \(26^4\)

(b) How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition is not allowed?

Solution: \(26 \cdot 25 \cdot 24 \cdot 23\)
(c) How many 4-letter words contain at least one repeated letter?

**Solution:** \[ 26^4 - 26 \cdot 25 \cdot 24 \cdot 23 \] (Use complement trick: Subtract the count with no repeated letters from the total count, \(26^4\).)

(d) How many 4-letter words contain the letter X?

**Solution:** \[ 26^4 - 25^4 \] (Use complement trick: There are \(26^4\) 4-letter words that do not contain the letter X. Subtracting this from the total count gives \(26^4 - 25^4\) words that contain the letter X.)

(e) How many 4-letter words consist of only the letters X and/or Y? (The words XXXX and YYY are included in this count.)

**Solution:** \[ 2^4 \] (There are 4 positions to fill with letters, with 2 choices (X or Y) for each position, giving \(2 \cdot 2 \cdot 2 \cdot 2 = 2^4\) choices altogether.)

### Fun Problem of the Week

Here is a fun problem that is independent of the material covered above, and requires virtually no prerequisites. It is from the 2011 UI Freshman Math Contest.

Let \(x\) be the number whose decimal expansion consists of the sequence of natural numbers written next to each other, i.e., \(x = 0.12345678910111213\ldots\).

(a) Determine the 2011th digit after the decimal point of \(x\).

(b) Prove that \(x\) is irrational.

**Solution:**

(a) The answer is \(7\); this is just a matter of keeping track of the positions at which the numbers 1, \ldots, 9, 10, \ldots, 99, 100, \ldots, 999, etc. fall.

(b) This is the more interesting part. It relies on the fact that rational numbers are exactly those whose decimal expansion is periodic from some point onwards. Thus, to prove that \(x\) is irrational, we need to show that the given decimal expansion is not ultimately periodic.

We argue by contradiction. Suppose \(x\) were ultimately periodic. Then \(x\) must be of the form

\[ x = 0.123\ldots BBBBB\ldots, \]

where \(B = b_1 b_2 \ldots b_p\) is the periodic “block”. Now, by the construction of \(x\), for any \(n\) the string 100\ldots 0 (a 1 followed by with \(n\) 0’s) must occur somewhere along decimal expansion of \(x\). Thus, \(x\) must also be of the form

\[ x = 0.123\ldots 100\ldots 0\underbrace{0\ldots 0}_{n}, \]

for any \(n\). Comparing (*) and (**), we see that this is only possible if the block \(B\) consists of all 0’s. However, the same reasoning with the digit 0 replaced by 9 yields that \(B\) must consist of all 9’s, a contradiction. Therefore, the decimal expansion of \(x\) is not periodic from some point onwards, so \(x\) must be irrational.