1. **Diophantine equations.** Diophantine equations are equations in integers, e.g., $x^n + y^n = z^n$.

Most problems involving Diophantine equations ask you to show that an equation has no integer solution, or that it has only the “obvious” solutions. Here are some common strategies to approach such problems:

- To prove the *non-existence* of a solution, use contradiction: Assume a solution exists and show that this leads to a contradiction.
- Simplify the equation by dividing out common factors on both sides.
- Consider parity (even/odd) or congruences to an appropriate modulus. If you can show that the left and right sides are always of opposite parity (i.e., one is even, the other one is odd), they can obviously not be equal. Similarly, if the left side is always congruent to 1 or 2 modulo 4, and the right side is always congruent to 3 modulo 4, the two sides can not be equal.

(a) Prove that a sum of two *odd* squares (i.e., squares of odd integers) can never be equal to a square of an integer.

(b) (2014 UI Mock Putnam Exam) Show that a positive integer whose decimal representation contains each of the digits 1, 2, 3, 4, 5, 6, 7 exactly 3 times and does *not* contain the digit 8 (but with no restrictions on the number of the digits 0 and 9) cannot be a perfect square (i.e., a square of an integer).

(c) Let $a, b, c$ be *odd* integers. Show that the equation $ax^2 + bx + c = 0$ has no integer solution $x$. (Hint: Do not use the quadratic formula...)

(d) Let $a, b, c$ be *odd* integers. Show that the equation $ax^2 + bx + c = 0$ has no rational solution $x$. (Hint: Convert the problem to one involving integer solutions.)
2. **Primes and composite numbers:** An integer $n \geq 2$ is called **composite** if there exist integers $a, b \in \{2, 3, 4, \ldots\}$ such that $n = ab$; it is called **prime** if it cannot be written in this form.

(a) (B1, Putnam 1988) Show that any composite integer is expressible as $xy + xz + yz + 1$ with positive integers $x, y, z$.

(Hint: By definition, a composite number be written in the form $n = ab$ with $a, b$ integers $\geq 2$. Thus, the problem reduces to showing that such a product $ab$ can always be written in the form $xy + xz + yz + 1$, with appropriate choices of $x, y, z$.)

(b) (A1, Putnam 1989) Find all prime numbers in the sequence 101, 10101, 1010101, 101010101, \ldots.

(Hint: Find a general formula for these numbers, then try to split this formula into two (nontrivial) integer factors.)

(c) Show that, for any base $b \geq 2$, the number $(10101)_b$ (i.e., 10101 interpreted in base $b$) is composite.

(Hint: Let $f(b)$ denote this number. Express $f(b)$ as a polynomial in $b$ and try to factor it.)

(a) **The missing digit in** $2^{29}$: The number $2^{29}$ is known to consist of exactly 9 decimal digits, all of which are pairwise distinct. Thus, exactly one of the ten digits 0, 1, 2, ..., 9 is missing. Without using a calculator or brute force hand calculation, determine which digit is missing.

(b) **A famous Diophantine equation**: Find all integer solutions to the equation $x^2 = 2y^2$.
   (Hint: This is a disguised form of a very famous result, with a classical proof!)

(c) **Primality of polynomial values.** Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree $n \geq 1$ with integer coefficients. Show that there exist infinitely many positive integers $n$ such that $|P(n)|$ is composite.
Challenge Problem of the Week

(B5, Putnam 1997) Let

$$a_n = 2^{2^{\cdots^{2}n}}$$

denote a “tower” of $n$ 2’s, with the order of the exponentiations from top to bottom. Thus, $a_1 = 2$, $a_2 = 2^2 = 4$, $a_3 = 2^{(2^2)} = 2^4 = 16$, etc. Prove that, for any integer $m \geq 2$, the sequence $a_n \mod m$, $n = 1, 2, 3, \ldots$, eventually becomes constant.

Happy Problemsolving!