UI Putnam Training Sessions, Beginner Level
Problem Set 3: Number Theory I  [Solutions]

http://www.math.illinois.edu/contests.html

The Problems

1. **Warmup: Practice with congruence notations:**
   
   (a) Which of the following congruences are true?
      
      (i) 37 ≡ −13 mod 3.
      
      **Solution:** \[ \text{\color{red}{F}} \]
      
      (ii) 107 ≡ 72 mod 7.
      
      **Solution:** \[ \text{\color{blue}{T}} \]
      
      (iii) \( n \equiv −n \mod 2 \) for any integer \( n \).
      
      **Solution:** \[ \text{\color{blue}{T}} \]
      
      (v) \( n \equiv 2n \mod 2 \) for any integer \( n \).
      
      **Solution:** \[ \text{\color{red}{F}} \]
      
      (vi) \( n(n + 1) \equiv 0 \mod 2 \) for any integer \( n \).
      
      **Solution:** \[ \text{\color{blue}{T}} \]
   
   (b) What is the remainder of \( 100 \cdot 101 \cdot 102 \cdot 103 \) when divided by 99? (This requires almost no computations if approached the “right” way! Don’t try to multiply out the numbers.)
   
   **Solution:** \[ \text{\color{blue}{24}} \]. To get this with almost no calculations, reduce each factor modulo 99 before multiplying: \( 100 \equiv 1 \mod 99 \), \( 101 \equiv 2 \mod 99 \), etc., so \( 100 \cdot 101 \cdot 102 \cdot 103 \equiv 1 \cdot 2 \cdot 3 \cdot 4 = 24 \mod 99 \).
   
   (c) What is the remainder of \( 100 \cdot 99 \cdot 98 \cdot 97 \) when divided by 101? (Again, don’t try to multiply out the numbers.)
   
   **Solution:** \[ \text{\color{blue}{24}} \]. Use the same idea as before: \( 100\cdot 99 \cdot 98 \cdot 97 \) is congruent to \( (−1)(−2)(−3)(−4) = 24 \mod 101 \).

2. **Congruence magic, I: Quick computation of remainders and last digits:** Use congruence magic to compute the following quickly and painlessly, *without multiplying large numbers*.

   (a) The remainder of \( 2013^{2014} \) when divided by 2014?
   
   **Solution:** \[ \text{\color{blue}{1}} \]. \( 2013 \equiv −1 \mod 2014 \), \( 2013^{2014} \equiv (−1)^{2014} = 1 \mod 2014 \).
   
   (b) The remainder of \( 1001^{1001} \) when divided by 3.
   
   **Solution:** \[ \text{\color{blue}{2}} \]. \( 1001 \equiv 2 \equiv −1 \mod 3 \), \( 2^{1001} \equiv (−1)^{1001} = −1 \equiv 2 \mod 3 \).
   
   (c) The last decimal digit of \( 3^{347} \).
   
   **Solution:** \[ \text{\color{blue}{7}} \]. We need to find the remainder of \( 3^{347} \mod 10 \). We proceed as follows: First, we find a small power of 3 that is congruent to 1 (or −1) modulo 10: This is easily done by trial and error:

   \[
   3^2 = 9 \equiv −1 \mod 10, \quad 3^4 = (3^2)^2 \equiv (−1)^2 = 1 \mod 10.
   \]
Next, we write the given exponent 347 as a multiple of this (small) exponent we have found plus a remainder:

$$347 = 4 \cdot 86 + 3.$$ 

Finally, we use modular arithmetic and the fact that $3^1 \equiv 1 \mod 10$ to find $3^{347} \mod 10$:

$$3^{347} = 3^{4\cdot 86 + 3} = (3^4)^{86} \cdot 3^3 \equiv 1^{86} \cdot 27 \equiv 7 \mod 10.$$ 

(d) The last two decimal digits of $99^{1001}$.

Solution: To get the last two digits, we need to work modulo 100: $99 \equiv -1 \mod 100, 99^{1001} \equiv (-1)^{1001} = -1 \equiv 99 \mod 100$.

(e) (UI Freshman Math Contest, 2012) Prove that there exists no power of 2 whose decimal representation ends in the digits 2012. (Hint: Consider congruences modulo 8.)

Solution: A number ending in 2012 must be of the form $n = 10000k + 2012$ for some integer $k$.

Since $10000 \equiv 0 \mod 8$ and $2012 = 8 \cdot 251 + 4 \equiv 4 \mod 8$, any such number must be congruent to 4 modulo 8. On the other hand, any power of 2 greater than 4 is congruent to 0 modulo 8, and hence cannot equal a number of the above form.

3. Congruence magic, II: Divisibility tests: Congruences can be used to justify, and generalize, the familiar divisibility tests by 3 and 9. This exercise guides you through the process.

(a) Given an integer $n$ with decimal expansion $n = (a_k a_{k-1} \ldots a_0)_{10}$, express $n$ as a sum involving powers of 10.

Solution: $n = (a_k a_{k-1} \ldots a_0)_{10} = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_0 10^0$.

(b) Now take this expression, reduce modulo 9, and simplify as much as possible. Show that the result is the sum of digits, $a_0 + a_1 + \cdots + a_k$.

In other words, if $s(n)$ denotes the sum of decimal digits of $n$, then we have $[s(n) \equiv n \mod 9]$. In particular, $n$ is divisible by 9 (i.e., $n \equiv 0 \mod 9$) if and only if $s(n)$ is divisible by 9 (i.e., $s(n) \equiv 0 \mod 9$). This is the divisibility test for 9.

Solution: Note that $10 \equiv 1 \mod 9$, and hence $10^i \equiv 1^i = 1 \mod 9$. Thus,

$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_0 10^0 \equiv a_k 1^k + a_{k-1} 1^{k-1} + \cdots + a_0 1^0 \mod 9 \equiv a_k + a_{k-1} + \cdots + a_0 \mod 9$$

which proves that $n$ is congruence to the sum of its digits, $a_k + a_{k-1} + \cdots + a_0$.

(c) An application: Show that any integer that contains each of the nine digits 1, 2, $\ldots$, 9 exactly once (for example 359261784) is divisible by 9.

Solution: The sum of digits of such an integer must be $1 + 2 + \cdots + 9 = 45$, and since 45 is divisible by 9, the integer itself must be divisible by 9.

(d) Another application. Let $n = \overline{4444} \ldots 4$ be a number consisting of 4444 4's when written in decimal. What is the remainder of $n$ when divided by 9?

Solution: $1$. By the above divisibility test, $n \equiv s(n) \mod 9$. Now, $s(n) = 4444 \cdot 4$, $4444 \equiv 4 + 4 + 4 + 4 = 16 \equiv 7 \mod 9$, so $s(n) \equiv 7 \cdot 4 = 28 \equiv 1 \mod 9$.

(e) Divisibility test for 11. Using a similar approach as above, find a divisibility test for 11. Then use this test to find the remainder of 123456789 when divided by 11.
Solution: Using the same argument as above, we get that, for \( n = (a_k a_{k-1} \ldots a_0)_{10} \),

\[
n \equiv a_0 - a_1 + a_2 - a_3 \ldots + (-1)^k a_k \mod 11.
\]

Thus, \( n \equiv t(n) \mod 11 \), where \( t(n) = a_0 - a_1 + a_2 - \ldots \) denotes the alternating sum of decimal digits of \( n \). For example, when \( n = 123456789 \) is the given number, then \( t(n) = 9 - 8 + 7 - 6 + 5 - 4 + 3 - 2 + 1 = 5 \), so the remainder of this number when divided by 11 is 5.

Fun Problem of the Week

The Four 4’s Problem (International Mathematical Olympiad, 1975). Let \( A = 44444444 \). Let \( B \) be the sum of the decimal digits of \( A \). Let \( C \) be the sum of the decimal digits of \( B \). Let \( D \) be the sum of the decimal digits of \( C \). What is \( D \)?

Solution: Since an integer is congruent to the sum of its digits modulo 9, the number \( D \) must have the same remainder modulo 9 as the given number \( A \). Now, \( A \) modulo 9 can easily be computed using a bit of congruence magic:

\[4444 \equiv 4 + 4 + 4 + 4 = 16 \equiv 7 \equiv -2 \mod 9,\]
\[(-2)^6 \equiv 64 \equiv 1 \mod 9,\]
\[4444^{4444} \equiv (-2)^{4444} \mod 9\]
\[= (-2)^{6 \cdot 740 + 4} \mod 9\]
\[\equiv 1^{740} \cdot (-2)^4 \mod 9\]
\[\equiv 16 \equiv 7 \mod 9\]

Thus, \( D \) must be congruent to 7 modulo 9, i.e., one of the numbers 7, 16, 25, \ldots

We now show that \( D \) is in fact equal to 7, by showing that (*) \( D \leq 15 \):

\[A = 4444^{4444} < (10^4)^{4444} < 10^{20,000},\]
\[B \leq 9 \cdot 20,000 = 180,000 < 10^6,\]
\[C \leq 9 \cdot 6 = 54,\]
\[D \leq 5 + 9 = 14,\]

since \( C \) has at most two digits and the first digit of \( C \) cannot be greater than 5. This proves (*) and completes the proof.

Happy Problemsolving!