The Problems

1. **Basic counting problems**: The following problems can all be solved using one of the basic counting formulas for the number of permutations, combinations, and subsets. For word counting problems, any string of letters counts as a “word”, and all words are assumed to be in upper case (i.e., capital letters). For example, with this interpretation there are exactly six 3-letter words that contain each of the letters HAL: HAL, HLA, AHL, ALA, LHA, LAH. Assume there are 26 letters in the alphabet.

   (a) How many 10-letter words consist of exactly 6 letters X and 4 letters Y?

   **Solution:** \( \binom{10}{6} \) \( \text{Pick 6 slots out of 10 for the letter X (} \binom{10}{6} \text{ ways to do this), then fill the remaining 4 slots with Y's (only one way).} \)

   (b) How many 10-letter words consist of only the letters X and/or Y (including the single-letter words X...X and Y...Y)?

   **Solution:** \( 2^{10} \) \( \text{(2 choices for each of the 10 slots.)} \)

   (c) How many 10-letter words consist of only the letters X and Y, with at least one letter of each type? (Here, the single letter words X...X and Y...Y are not counted.)

   **Solution:** \( 2^{10} - 2 \)

   (d) How many 10-letter words have exactly two distinct letters?

   **Solution:** \( \binom{26}{2}(2^{10} - 2) \) \( \text{Pick 2 letters out of 26, then form a word with these two letters.)} \)

   (e) How many 4-letter words contain no repeated letter and have all letters occurring in alphabetically increasing order (e.g., AFGL is counted, but not FAGL, since F and A are not in alphabetically increasing order). (Hint: If approached the right way, this has a very simple answer. Don’t attempt brute force counting!)

   **Solution:** \( \binom{26}{4} \) \( \text{(Pick 4 distinct letters out of 26 (} \binom{26}{4} \text{ choices), then put them in alphabetical order (only one way to do this).)} \)

2. **COOL TRICK 1: The MISSISSIPPI formula.**

   How many ways are there to arrange the letters of MISSISSIPPI in some order? I.e., how many 11-letter words can be formed with the letters of MISSISSIPPI?

   The following steps guide you through the derivation of this formula:

   (a) How many 4-letter words can be formed with the letters M,A,T,H?

   **Solution:** \( 4! \)

   (b) How many 4-letter words can be formed with the letters M,A,T,T? 

   Note that this time two of the letters are the same. How does this affect the result?

   **Solution:** \( 4!/2 \)
(c) How many 5-letter words can be formed with the letters M,A,T,T,T?

This is a similar, but more complicated, situation, with 2 non-repeated letters (M,A) and 3 T’s. To analyze this situation, proceed as follows:

- Assume first the 3 T’s are distinct letters, by labeling them T₁, T₂, T₃. What is the total count under this assumption? i.e., the number of 5-letter words formed by the letters M,A,T₁,T₂,T₃?

**Solution:** \(5!\)

- Now, remove the subscripts in the T’s, and try to match subscripted words with non-subscripted words; for example:

  \[ ATMTT \leftrightarrow AT₁MT₂T₃, \ AT₁MT₃T₂, \ AT₂MT₁T₃, \]
  \[ AT₂MT₃T₁, \ AT₃MT₁T₂, \ AT₃MT₂T₁, \]

  How many subscripted words correspond to each non-subscripted word?

**Solution:** \(3!\)

- Using the results of the previous two parts, derive answer the original question, i.e., find the total number of 5-letter words that can be formed with the letters M,A,T,T,T.

**Solution:** \(\frac{5!}{3!}\)

(d) Now derive the “MISSISSIPPI formula”: Find the number of words that can be formed with the 11 letters M,I,S,S,I,S,S,I,P,P,I (4 I’s, 4 S’s, 2 P’s, 1 M).

**Solution:** \(\frac{11!}{4!4!2!1!}\)

3. COOL TRICK 2: The donut counting formula and the star/bar technique

The following steps guide you through the derivation of this formula.

- Imagine the donuts lined up left to right with dividers between the different varieties, with the 1st type (plain) on the left, followed by the second type (chocolate), then the third type (glazed), and finally the fourth type (pumpkin). Note that exactly 3 dividers are needed to separate the 4 types.

- Given a selection of 10 donuts from these 4 varieties, line these donuts up in the same manner, with dividers separating the varieties. For example, a selection of 4 donuts of type 1, 3 of type 2, 0 of type 3, and 3 of type 4, would be represented as follows:

  \((\ast)\)
  \[ oooo \mid ooo \mid ooo \]

  (Using star/bar notation this could also be represented as

  \[ * * * \mid * * \mid * * * \]

  but o’s instead of *’s are more suggestive of donuts.)

- Now interpret \((\ast)\) as a binary string formed with the symbols o and |. The number of o’s in \((\ast)\) is equal to the total number of donuts, i.e., 10 in our case; the number of |’s is equal to the number of varieties minus 1, i.e., 4 − 1 in our case.

- Altogether, \((\ast)\) gives an encoding of our donut selection by a 13-digit binary string consisting of 10 0’s and 3 1’s. It is easy to see that every such binary string encodes a unique donut selection. Thus, the total number of donut selections is equal to the total number of 13-digit binary strings with 10 0’s and 3 1’s.

- The number of such binary strings is equal to the number of ways to pick 3 spots out for 13 for the 1’s, i.e., \(\binom{13}{3}\).
• The same argument shows that the number of ways to select \( n \) donuts from \( r \) varieties is 
\[
\binom{n + r - 1}{r - 1}
\]  
This is the “donut counting formula.”

Here are some applications of this formula:

(a) In how many ways can 10 be written as a sum of 4 nonnegative integers, if the order is taken into account (so that, for example, \( 10 = 3 + 2 + 4 + 1 \) and \( 10 = 3 + 4 + 2 + 1 \) count as different representations)?

Solution: \( \binom{10 + 3}{3} \) This is equivalent to the above donut counting problem: A donut selection can be encoded as a tuple \( (n_1, n_2, n_3, n_4) \) where the \( n_i \)'s denote the number of donuts chosen from the 4 varieties. The \( n_i \)'s have to be nonnegative integers, and they must add up to the total number, 10, of donuts, but are otherwise unrestricted. Thus, the number of possible donut selections is the same as the number of tuples of 4 nonnegative integers with sum 10.

(b) How many ways are there to select 10 donuts from 4 varieties if you are required to choose at least 1 donut from each variety? (Hint: Start out by taking the required 1 donut from each of the 4 varieties, then pick the remaining 6 ...)

Solution: \( \binom{6 + 3}{3} \). After taking the “required” 4 donuts out, we have 6 donuts left to pick from 4 varieties, and applying the donut formula gives the answer.

(c) Encode the 26 letters in the alphabet by the numbers 0, 1, \ldots, 25 so that A has code 0, B has code 1, etc., and Z has code 25. Define the “code” of a word to be the sum of the codes of its letters. How many 9-letter words have code 25?

Solution: \( \binom{25 + 8}{8} \). This is equal to the number of ways 25 can be written as a sum of 9 nonnegative numbers, and the donut-counting formula applies.

4. Intermediate/harder counting problems. The following problems all have simple answers (often just a single binomial coefficient), but require some clever thinking. For example, you may need to find an appropriate encoding to connect the given problem with one of the standard counting formulas.

(a) **Sequence counting:** How many sequences of 1’s and \(-1\)'s of length 10 are there that sum up to 2?

Solution: \( \binom{10}{6} \) To get a sum of 2, 6 of the 10 elements in the sequence must be 1’s and 4 must be \(-1\)’s; the number of binary sequences with 6 1’s and 4 \(-1\)’s is \( \binom{10}{6} \).

(b) **Path counting, I:** Imagine a walk in the plane with two possible moves, U (up by one unit), R (right by one unit). How many ways are there to get from the point \((0, 0)\) to the point \((a, b)\) (where \(a, b\) are nonnegative integers)?

Solution: \( \binom{a+b}{a} \) Encode each path as a word formed with the letters R and U. To reach the point \((a, b)\) requires exactly \( a \) R moves and \( b \) U moves, so the paths we seek to count correspond to words of length \( a + b \) consisting of \( a \) R’s and \( b \) U’s. There are \( \binom{a+b}{a} \) such words.

(c) **Path counting, II:** Now consider a walk in the plane with four possible moves, given by the four vectors \((\pm 1, \pm 1)\). How many ways are there to get from the origin back to the origin in 2n moves?

Solution: \( \binom{2n}{n}^2 \) In order to get back to the origin, there must be an equal number of +1’s and \(-1\)’s in each of the two coordinates of the 2n move vectors \((\pm 1, \pm 1)\). There are \( \binom{2n}{n} \) ways to place an equal number of +1’s and \(-1\)’s into the first coordinates of the 2n vectors, and similarly \( \binom{2n}{n} \) ways to fill the second coordinates of these vectors with an equal number of +1’s and \(-1\)’s. Thus the total number of such vectors is \( \binom{2n}{n}^2 \).

(d) **A geometric application:** Counting intersections of diagonals in an \( n \)-gon. Assume \( n \) points are arranged around a circle and all chords (i.e., diagonals) are drawn. How many points of intersection between these chords are there, assuming no three chords intersect at the same point? (At first glance
this seems to lead to messy summations, but the answer turns out to be very simple, and it can be found with some clever thinking and zero calculations.)

Solution: \[ \binom{n}{3} \] Observe that any 4 points on the circle generate exactly one intersection point; conversely, any intersection point corresponds to exactly 4 points on the circle, namely the two pairs of endpoints of the chords passing through this point. Hence the number of intersection points is equal to the number of ways one can choose 4 points out of the n points on the circle, i.e., \( \binom{n}{4} \).

(e) Counting disjoint subset pairs: How many ordered pairs of subsets \((A, B)\) of \(\{1, 2, \ldots, 10\}\) are there such that \(A \cap B = \emptyset\)? (Hint: Think of the standard encoding of subsets via binary strings and find a similar encoding for pairs of subsets of the above type.)

Solution: \[ \frac{3^{10}}{2} \] Encode each such pair \((A, B)\) by a 10-letter code word, whose \(i\)-th letter is \(A\) if the number \(i\) belongs to \(A\), \(B\) if \(i\) belongs to \(B\), and \(O\) otherwise. (The disjointness condition ensures that exactly one of those three cases must occur, so the encoding is well defined.) For example, the pair \((\{1, 3\}, \{5, 6, 7, 8\})\) is encoded by the word AOAOBBBOO. Clearly, this encoding is a \(1 - 1\) correspondence between pairs \((A, B)\) of subsets of the given type and 10-letter words formed from the 3 letters \(A, B, O\). Since there are \(3^{10}\) such words, the total number of pairs \((A, B)\) must also be \(3^{10}\).

(f) Numbers with prescribed prime factors: How many positive integers are there that contain no repeated prime factor and no prime factor greater than 13? (Thus, the sequence of these numbers is \(1, 2, 3, 5, 2 \cdot 3, \ldots, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13\).)

Solution: \[ 2^6 \] The numbers we need to count are those of the form \(2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}13^{a_6}\), with exponents \(a_i = 0, 1\). There are \(2^6\) tuples \((a_1, \ldots, a_6)\) of such exponents \(a_i\), and hence \(2^6\) numbers of the desired form.

(g) Pairs of coprime numbers with prescribed prime factors: How many order pairs \((a, b)\) of positive integers with no repeated prime factor and no prime factor greater than 13 are there such that \(a\) and \(b\) have no common prime factor? (For example, \((6, 15)\) is not counted since 6 and 15 have a common prime factor, but \((6, 35)\) is counted? (Hint: Encode!)

Solution: \[ 3^6 \] This is similar to (e). Each of the 6 primes \(2, \ldots, 13\) can be a factor of \(a\) but not \(b\), a factor of \(b\) but not \(a\), or a factor of neither \(a\) nor \(b\), giving a total of 3 choices for each prime, and an overall count of \(3^6\).

Challenge Problem of the Week

Call a sequence \(a_1, a_2, \ldots, a_n\) of distinct real numbers a record sequence if for each \(i = 2, 3, \ldots, n\) we have either \(a_i > a_j\) for all \(j < i\) (so that \(a_i\) is a “record high”) or \(a_i < a_j\) for all \(j < i\) (so that \(a_i\) is a “record low”). For example, the sequence 1, 2, 3, 4, 5 is a a record sequence, as is the sequence 4, 3, 2, 5, 1.

| How many permutations of \(1, 2, \ldots, n\) are record sequences in the above sense? |

Solution: The answer is \(2^{n-1}\).

Proof: We need to count the number of those permutations that represent a record sequence in the above sense. The trick is to work backwards, starting with the last element in the sequence. There are exactly 2 choices for the last (i.e., \(n\)th) element of the sequence: it can be either the largest or the smallest among the \(n\) numbers \(1, \ldots, n\). Once we have chosen the \(n\)th element, we have exactly 2 choices for the \((n-1)\)th element, namely the smallest or the largest of the remaining \(n-1\) numbers. Continuing in this manner down to the 2nd element, we see that there are a total of \(2^{n-1}\) ways to place the elements 2 through \(n\) in the sequence, and one way to place the remaining element. Hence, the number of record permutations is \(2^{n-1}\).