UI Putnam Training Sessions, Advanced Level
Problem Set 2: Advanced Counting

http://www.math.illinois.edu/contests.html

Tips on approaching counting problems

• **DO NOT USE CALCULATORS!** All problems are intended to be done without a calculator; leave answers in “raw” form such as $36 \cdot 35 \cdot 16$, or $4^{20}$, or $26!$. You won’t be able to use calculators in any of our math contests, so try to get into the habit of doing things without a calculator.

• **Know the basic counting formulas:**
  - Number of permutations of $n$ objects: $n!$.
  - Number of subsets of an $n$-element set: $2^n$.
  - Number of $k$-element subsets of an $n$-element set: \( \binom{n}{k} \).

**NOTE:** Always use the “choose” notation, \( \binom{n}{k} \), for binomial coefficients; avoid $C$-notations like \( C^n_k \), etc.

• **Try to find the simplest answer; avoid messy summation formulas.** All problems in this set have a simple answer, usually consisting of just one or two terms; try to find this answer! A lengthy summation may be technically correct, but would defeat the purpose of the problem and would be useless without a calculator. For example, the number of nonempty subsets of a 100-element set is $2^{100} - 1$ (the “simple” answer), though it could also be expressed as the sum $\sum_{k=1}^{100} \binom{100}{k}$ (a “messy” and useless answer).

• **Break down the counting process into independent stages if possible:** For example, for word counting problems try to construct the words you want to count by filling in slots for letters one at a time, from left to right, count the number of choices you have for each slot, and multiply these counts to get the overall count. This works whenever the stages are independent.

• **Count the complement:** For some problems the complement trick works wonders: Instead of counting directly the number of objects (words, strings, numbers, etc.) with the desired property, try to count the number of objects that do not have this property, and then subtract this count from the overall count.

• **Try encoding:** A powerful method is to encode the objects to be counted in a form that is easier to count, e.g., as binary sequences or “words”. For example, subsets of an $n$-element set \( \{1, 2, \ldots, n\} \) have a natural encoding as binary strings of length $n$, with the elements of the subset corresponding to the positions of the 1’s in the string: 0100 $\leftrightarrow \{2\}$, 0110 $\leftrightarrow \{2, 3\}$, etc. With this encoding, the total number of subsets of an $n$-is equal to the total number of binary strings of length $n$, which is $2^n$. (Mathematically, the encoding gives a bijection between the subsets of an $n$-element set and the binary strings of length $n$.)

• **Special tricks/formulas:** Two ingenious tricks/formulas that are worth knowing are the **MISSISSIPPI formula** and the **donut counting formula**. See the problems for details, derivations, and examples.
The Problems

1. **Basic counting problems:** The following problems can all be solved using one of the basic counting formulas for the number of permutations, combinations, and subsets. For word counting problems, any string of letters counts as a “word”, and all words are assumed to be in upper case (i.e., capital letters). For example, with this interpretation there are exactly six 3-letter words that contain each of the letters HAL: HAL, HLA, AHL, ALA, LHA, LAH. Assume there are 26 letters in the alphabet.

   (a) How many 10-letter words consist of exactly 6 letters X and 4 letters Y?
   (b) How many 10-letter words consist of only the letters X and/or Y (including the single-letter words X...X and Y...Y)?
   (c) How many 10-letter words consist of only the letters X and Y, with at least one letter of each type? (Here, the single letter words X...X and Y...Y are not counted.)
   (d) How many 10-letter words have exactly two distinct letters?
   (e) How many 4-letter words contain no repeated letter and have all letters occurring in *alphabetically increasing order* (e.g., AFGL is counted, but not FAGL, since F and A are not in alphabetically increasing order). (Hint: If approached the right way, this has a very simple answer. Don’t attempt brute force counting!)
2. **COOL TRICK 1: The MISSISSIPPI formula.**

How many ways are there to arrange the letters of MISSISSIPPI in some order? I.e., how many 11-letter words can be formed with the letters of MISSISSIPPI?

The following steps guide you through the derivation of this formula:

(a) How many 4-letter words can be formed with the letters M,A,T,H?

(b) How many 4-letter words can be formed with the letters M,A,T,T?

*Note that this time two of the letters are the same. How does this affect the result?*

(c) How many 5-letter words can be formed with the letters M,A,T,T,T?

This is a similar, but more complicated, situation, with 2 non-repeated letters (M,A) and 3 T’s. To analyze this situation, proceed as follows:

- Assume first the 3 T’s are distinct letters, by labeling them T₁, T₂, T₃. What is the total count under this assumption? i.e., the number of 5-letter words formed by the letters M,A,T₁,T₂,T₃?
- Now, remove the subscripts in the T’s, and try to match subscripted words with non-subscripted words; for example:

  \[ \text{ATMTT} \leftrightarrow \text{AT₁MT₂T₃}, \text{AT₁MT₃T₂}, \text{AT₂MT₁T₃}, \text{AT₂MT₃T₁}, \text{AT₃MT₁T₂}, \text{AT₃MT₂T₁}, \]

  How many subscripted words correspond to each non-subscripted word?

- Using the results of the previous two parts, derive answer the original question, i.e., find the total number of 5-letter words that can be formed with the letters M,A,T,T,T.

(d) Now derive the “MISSISSIPPI formula”: Find the number of words that can be formed with the 11 letters M,I,S,S,I,S,S,I,P,P,I (4 I’s, 4 S’s, 2 P’s, 1 M).
3. COOL TRICK 2: The donut counting formula and the star/bar technique

How many ways are there to place an order of 10 donuts if there are 4 varieties to choose from? (For example, if the varieties are plain, chocolate, glazed, pumpkin, a possible order might be 4 plain, 3 chocolate, 0 glazed, and 3 pumpkin donuts.)

The following steps guide you through the derivation of this formula.

- Imagine the donuts lined up left to right with dividers between the different varieties, with the 1st type (plain) on the left, followed by the second type (chocolate), then the third type (glazed), and finally the fourth type (pumpkin). Note that exactly 3 dividers are needed to separate the 4 types.
- Given a selection of 10 donuts from these 4 varieties, line these donuts up in the same manner, with dividers separating the varieties. For example, a selection of 4 donuts of type 1, 3 of type 2, 0 of type 3, and 3 of type 4, would be represented as follows:

  \[
  (\ast) \quad \text{o}o0\text{01234567890}\text{o}000
  \]

  (Using star/bar notation this could also be represented as
  \[
  \ast\ast\ast\ast\mid\ast\ast\ast\mid\ast\ast\ast
  \]
  but o’s instead of *’s are more suggestive of donuts.)
- Now interpret (\ast) as a binary string formed with the symbols o and |. The number of o’s in (\ast) is equal to the total number of donuts, i.e., 10 in our case; the number of |’s is equal to the number of varieties minus 1, i.e., 4 – 1 in our case.
- Altogether, (\ast) gives an encoding of our donut selection by a 13-digit binary string consisting of 10 0’s and 3 1’s. It is easy to see that every such binary string encodes a unique donut selection. Thus, the total number of donut selections is equal to the total number of 13-digit binary strings with 10 0’s and 3 1’s.
- The number of such binary strings is equal to the number of ways to pick 3 spots out for 13 for the 1’s, i.e., \(\binom{13}{3}\).
- The same argument shows that the number of ways to select \(n\) donuts from \(r\) varieties is \(\binom{n + r - 1}{r - 1}\). This is the “donut counting formula.”

Here are some applications of this formula:

(a) In how many ways can 10 be written as a sum of 4 nonnegative integers, if the order is taken into account (so that, for example, 10 = 3 + 2 + 4 + 1 and 10 = 3 + 4 + 2 + 1 count as different representations)?

(b) How many ways are there to select 10 donuts from 4 varieties if you are required to choose at least 1 donut from each variety? (Hint: Start out by taking the required 1 donut from each of the 4 varieties, then pick the remaining 6 ...)

(c) Encode the 26 letters in the alphabet by the numbers 0, 1, . . . , 25 so that A has code 0, B has code 1, etc., and Z has code 25. Define the “code” of a word to be the sum of the codes of its letters. How many 9-letter words have code 25?
4. **Intermediate/harder counting problems.** The following problems all have simple answers (often just a single binomial coefficient), but require some clever thinking. For example, you may need to find an appropriate encoding to connect the given problem with one of the standard counting formulas.

(a) **Sequence counting:** How many sequences of 1’s and −1’s of length 10 are there that sum up to 2?

(b) **Path counting, I:** Imagine a walk in the plane with two possible moves, U (up by one unit), R (right by one unit). How many ways are there to get from the point (0, 0) to the point (a, b) (where a, b are nonnegative integers)?

(c) **Path counting, II:** Now consider a walk in the plane with four possible moves, given by the four vectors (±1, ±1). How many ways are there to get from the origin back to the origin in 2n moves?

(d) **A geometric application:** Counting intersections of diagonals in an n-gon. Assume n points are arranged around a circle and all chords (i.e., diagonals) are drawn. How many points of intersection between these chords are there, assuming no three chords intersect at the same point? (At first glance this seems to lead to messy summations, but the answer turns out to be very simple, and it can be found with some clever thinking and zero calculations.)

(e) **Counting disjoint subset pairs:** How many ordered pairs of subsets (A, B) of \{1, 2, … , 10\} are there such that \( A \cap B = \emptyset \)? (Hint: Think of the standard encoding of subsets via binary strings and find a similar encoding for pairs of subsets of the above type.)

(f) **Numbers with prescribed prime factors:** How many positive integers are there that contain no repeated prime factor and no prime factor greater than 13? (Thus, the sequence of these numbers is 1, 2, 3, 5, 2 · 3, … , 2 · 3 · 5 · 7 · 11 · 13.)

(g) **Pairs of coprime numbers with prescribed prime factors:** How many order pairs \((a, b)\) of positive integers with no repeated prime factor and no prime factor greater than 13 are there such that \(a\) and \(b\) have no common prime factor? (For example, (6, 15) is not counted since 6 and 15 have a common prime factor, but (6, 35) is counted? (Hint: Encode!)
Challenge Problem of the Week

Call a sequence $a_1, a_2, \ldots, a_n$ of distinct real numbers a **record sequence** if for each $i = 2, 3, \ldots, n$ we have either $a_i > a_j$ for all $j < i$ (so that $a_i$ is a “record high”) or $a_i < a_j$ for all $j < i$ (so that $a_i$ is a “record low”). For example, the sequence $1, 2, 3, 4, 5$ is a record sequence, as is the sequence $4, 3, 2, 5, 1$.

How many permutations of $1, 2, \ldots, n$ are record sequences in the above sense?