2022 UI UNDERGRADUATE MATH CONTEST

March 26, 2022, 1 pm – 4 pm

Solutions

1. (Variation of Problem B2, Putnam 1983) Determine, with proof, the number of ways 2022 can be written as a sum of nonnegative powers of 2 such that each power of 2 can be used at most 3 times. (For example, the number 8 can be written in 5 different ways in this manner: $8 = 8$, $8 = 4 + 4$, $8 = 4 + 2 + 2$, $8 = 4 + 2 + 1 + 1$, $8 = 2 + 2 + 2 + 1 + 1$.)

Solution. More generally, let $f(n)$ denote the number of representations of n as a sum of powers of 2, where each power of 2 can be used at most 3 times. We will show that

$$f(n) = \lfloor n/2 \rfloor + 1. \tag{1}$$

Thus $f(2022) = \boxed{1012}$.

To prove (1), note that a representation $n = \sum_{i=0}^{k} a_i 2^i$ with $a_i \in \{0, 1, 2, 3\}$ can be written uniquely in the form $n = \sum_{i=0}^{k} (2\delta_i + \epsilon_i) 2^i$ with $\delta_i, \epsilon_i \in \{0, 1\}$. Equivalently, we can write this representation as $n = 2n_1 + n_2$, where $n_1 = \sum_{i=0}^{k} \delta_i 2^i$ and $n_2 = \sum_{i=0}^{k} \epsilon_i 2^i$ are nonnegative integers. Conversely, each pair (n_1, n_2) of nonnegative integers such that $n = 2n_1 + n_2$ yields a representation of n of the desired form by letting $a_i = 2\delta_i + \epsilon_i$, where δ_i and ϵ_i are the binary digits of n_1 and n_2. Thus, $f(n)$ is equal to the number of ways to write n in the form $n = 2n_1 + n_2$ with nonnegative integers n_1 and n_2. There are $\lfloor n/2 \rfloor + 1$ ways to choose n_1 and each such choice yields exactly one representation of the form $n = 2n_1 + n_2$, so we obtain $f(n) = \lfloor n/2 \rfloor + 1$, as desired.

2. Let n and k be integers with $1 \le k \le n - 1$. Prove that any prime power p^s that divides the binomial coefficient $\binom{n}{k}$ must satisfy $p^s \le n$.

Solution. Given a prime p, let $\nu_p(n)$ denote the exponent of p in n. Then

$$\nu_p\left(\binom{n}{p}\right) = \nu_p\left(\frac{n!}{k!(n-k)!}\right) = \nu_p(n!) - \nu_p(k!) - \nu_p((n-k)!). \tag{1}$$

Using the formula $\nu_p(m!) = \sum_{i=0}^{\infty} \lfloor m/p^i \rfloor$ (note that the sum here has only finitely many nonzero terms, so convergence is not an issue), we get

$$\nu_p\left(\binom{n}{p}\right) = \sum_{i=0}^{\infty} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor \right). \tag{2}$$

The $i = 0$ term in the sum (2) is $n - k - (n - k) = 0$. Moreover, the terms with $i > \log n / \log p$ are also 0 since each of the floor functions is 0 in this case. Thus the only nonzero terms in the sum (2) are those corresponding to $i = 1, 2, \ldots, \lfloor \log n / \log k \rfloor$. Each of these terms is of the form $\lfloor x \rfloor - \lfloor y \rfloor - \lfloor x - y \rfloor$, where $0 \le y \le x$. Since

$$\lfloor x \rfloor - \lfloor y \rfloor - \lfloor x - y \rfloor < x - (y - 1) - (x - y - 1) = 2,$$

and $\lfloor x \rfloor - \lfloor y \rfloor - \lfloor x - y \rfloor$ is an integer, each term on the right-hand side of (2) is bounded above by 1. Since there are at most $\lfloor \log n / \log p \rfloor$ nonzero terms, it follows that $\nu_p(\binom{n}{p}) \le \lfloor \log n / \log p \rfloor$. Hence the largest power p^s dividing $\binom{n}{k}$ must satisfy $p^s \le p^{\lfloor \log n / \log p \rfloor} \le n$, as claimed.

3. Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be an $n \times n$ matrix whose entries a_{ij} are nonnegative integers. Let $s_i = \sum_{j=1}^n a_{ij}$ and $t_j = \sum_{i=1}^n a_{ij}$ denote, respectively, the sums of the ith row and jth column of A. Suppose that $s_i + t_j \geq n$ for any pair (i, j) of indices for which $a_{ij} = 0$.

   Show that $\sum_{i,j=1}^n a_{ij} \geq n^2/2$.

   Solution. Let m denote the minimal value among all row and column sums s_i and t_j. If $m \geq n/2$, then $\sum_{i,j=1}^n a_{ij} = \sum_{i=1}^n s_i \geq n(n/2) = n^2/2$, as claimed.
   Now suppose that $m < n/2$. By rearranging rows and columns and transposing the matrix if necessary, we may assume that the minimum m is attained at the first row, i.e., that $m = s_1 = \sum_{j=1}^n a_{1j}$. Let J denote the set of indices j such that $a_{1j} = 0$. Since the entries a_{1j} are nonnegative integers with sum m, at most m entries can be nonzero, so we must have $|J| \geq n - m$, and since $m < n/2$, it follows that $|J| > n/2 > m$. Since $a_{1j} = 0$ for $j \in J$, we have, for each $j \in J$, $s_1 + t_j \geq n$ and therefore $t_j \geq n - s_1 = n - m$. On the other hand, by the minimality of m, we have $t_j \geq m$ for any index j. It follows that

$$\sum_{i,j=1}^n a_{ij} = \sum_{j=1}^n t_j = \sum_{j \in J} t_j + \sum_{j \notin J} t_j \geq |J|(n-m) + (n - |J|)m = |J|(n-2m) + nm.$$

   Since $|J| \geq n - m$ and $m < n/2$, we have

$$|J|(n-2m) + nm \geq (n-m)(n-2m) + nm = (n-m)^2 + m^2 \geq \frac{((n-m)+m)^2}{2} = \frac{n^2}{2},$$

   where the second-last step follows from Cauchy's inequality. This proves the desired bound.

4. Prove that the limit $\lim_{n\to\infty} n \sin(2\pi n! e)$ (where e is Euler's constant) exists and find its value.

   Solution. We have

$$\begin{aligned} n!e = n! \sum_{k=0}^{\infty} \frac{1}{k!} &= \sum_{k=0}^{n} \frac{n!}{k!} + \frac{1}{n+1} + \sum_{i=2}^{\infty} \frac{1}{(n+1)\ldots(n+i)} \\ &= a_n + \frac{1}{n+1} + R_n, \end{aligned}$$

   where a_n is an integer and R_n is a remainder satisfying

$$|R_n| \leq \frac{1}{(n+1)(n+2)} \sum_{j=0}^{\infty} \frac{1}{(n+2)^j} \leq \frac{2}{(n+1)(n+2)}.$$

   It follows that

$$n \sin(2\pi n! e) = n \sin\left(\frac{2\pi}{n+1} + O\left(\frac{1}{n^2}\right)\right)$$

   Since $\lim_{x\to 0} (\sin x)/x = 1$, the last expression converges as $n \to \infty$, with limit

$$\lim_{n\to\infty} n \sin(2\pi n! e) = \lim_{n\to\infty} \frac{2\pi n}{n+1} = \boxed{2\pi}.$$

5. Find, with proof, all pairs of function (f, g) from $\mathbb{R}$ to $\mathbb{R}$ that satisfy

$$f(x) - f(y) = (x - y)(g(x) + g(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

Solution. We will show that the pairs (f, g) satisying the given functional equation are exactly the functions of the form

$$(f(x), g(x)) = (ax^2 + 2bx + c, ax + b) \tag{1}$$

Applying the given functional equation with the variable pairs (x, y), (y, z), and (z, x) and adding the resulting equations gives

$$\begin{aligned} 0 &= (f(x) - f(y)) + (f(y) - f(z)) + (f(z) - f(x)) \\ &= (x - y)(g(x) + g(y)) + (y - z)(g(y) + g(z)) + (z - x)(g(z) + g(x)) \\ &= g(x)(z - y) + g(y)(x - z) + g(z)(y - x). \end{aligned} \tag{2}$$

Let $y = 1$ and $z = 0$ in (2). Then $0 = -g(x)+g(1)x+g(0)(1-x)$ and thus $g(x) = (g(1)-g(0))x+g(0)$. Thus, $g(x)$ is of the desired form $g(x) = ax + b$ in (1) with constants $b = g(0)$ and $a = g(1) - g(0)$.

Next, let $y = 0$ in the given functional equation and substitute $g(x) = ax + b$ and $g(0) = b$ on the right side. Then $f(x) - f(0) = x(ax + b + b) = ax^2 + 2bx$ and hence $f(x) = ax^2 + 2bx + c$, where $c = f(0)$. Thus $f(x)$ is also of the required form in (1).

We have thus shown that any pair (f, g) of functions satisfying the given functional equation must necessarily be of the form (1). For the converse, suppose functions f and g of the form (1). Then $f(x) - f(y) = a(x^2 - y^2) + 2b(x - y)$ and $(x - y)(g(x) + g(y)) = (x - y)(a(x + y) + 2b) = a(x^2 - y^2) + 2b(x - y)$, so f and g satisfy the given functional equation.

6. (Variation of Problem A3, Putnam 1966) Let $a_1, a_2, \dots$ be positive real numbers, and let $b_n = (1/n)\sum_{i=1}^n a_i$ be the arithmetic mean of the numbers $a_1, \dots, a_n$. Show that if the series $\sum_{n=1}^\infty 1/a_n$ converges, then so does the series $\sum_{n=1}^\infty 1/b_n$.

Solution. Let $c_1 \le c_2 \le \dots$ be the numbers a_i in increasing order. Such a re-ordering is possible since the convergence of $\sum_n 1/a_n$ implies that $a_n \to \infty$ as $n \to \infty$.

Since the terms $1/a_n$ are positive, re-ordering the series $\sum_{n=1}^\infty 1/a_n$ does not change its convergence, so the series $\sum_{n=1}^\infty 1/c_n$ converges as well. Moreover, since the numbers c_n are nondecreasing, we have, for $n \ge 2$,

$$nb_n = \sum_{i=1}^n a_i \ge \sum_{i=1}^n c_i \ge \lfloor n/2 \rfloor c_{\lfloor n/2 \rfloor} \ge \frac{n}{4} c_{\lfloor n/2 \rfloor}$$

Hence

$$\sum_{n=2}^\infty \frac{1}{b_n} \le 4 \sum_{n=2}^\infty \frac{1}{c_{\lfloor n/2 \rfloor}} = 8 \sum_{n=2}^\infty \frac{1}{c_n} < \infty$$

Thus the series $\sum_{n=1}^\infty 1/b_n$ converges.