2022 UI UNDERGRADUATE MATH CONTEST

March 26, 2022, 1 pm – 4 pm

1. Determine, with proof, the number of ways 2022 can be written as a sum of nonnegative powers of 2 such that each power of 2 can be used at most 3 times. (For example, the number 8 can be written in 5 different ways in this manner: $8 = 8$, $8 = 4+4$, $8 = 4+2+2$, $8 = 4+2+1+1$, $8 = 2+2+2+1+1$.)

2. Let n and k be integers with $1 \le k \le n-1$. Prove that any prime power p^s that divides the binomial coefficient $\binom{n}{k}$ must satisfy $p^s \le n$.

3. Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be an $n \times n$ matrix whose entries a_{ij} are nonnegative integers. Let $s_i = \sum_{j=1}^{n} a_{ij}$ and $t_j = \sum_{i=1}^{n} a_{ij}$ denote, respectively, the sums of the ith row and jth column of A. Suppose that $s_i + t_j \ge n$ for any pair (i, j) of indices for which $a_{ij} = 0$.

   Show that $\sum_{i,j=1}^{n} a_{ij} \ge n^2/2$.

4. Prove that the limit $\lim_{n\to\infty} n \sin(2\pi n! e)$ (where e is Euler's constant) exists and find its value.

5. Find, with proof, all pairs of function (f, g) from $\mathbb{R}$ to $\mathbb{R}$ that satisfy

$$f(x) - f(y) = (x - y)(g(x) + g(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

6. Let $a_1, a_2, \ldots$ be positive real numbers, and let $b_n = (1/n)\sum_{i=1}^{n} a_i$ be the arithmetic mean of the numbers $a_1, \ldots, a_n$. Show that if the series $\sum_{n=1}^{\infty} 1/a_n$ converges, then so does the series $\sum_{n=1}^{\infty} 1/b_n$.