# 2021 UI UNDERGRADUATE MATH CONTEST

## Solutions

1. Given a natural number $a_1$, define a sequence $\{a_n\}$ recursively by

$$a_{n+1} = 2a_n - \lfloor \sqrt{a_n} \rfloor^2 \quad (n = 1, 2, \dots),$$

where $\lfloor t \rfloor$ denotes the floor function, i.e., the greatest integer $\leq t$. For example, if $a_1 = 6$, then $a_2 = 2 \cdot 6 - 2^2 = 8$, $a_3 = 2 \cdot 8 - 2^2 = 12$, $a_4 = 2 \cdot 12 - 3^2 = 15$, and so on. Determine, with proof, all natural numbers $a_1$ for which the sequence $\{a_n\}$ eventually becomes constant.

**Solution.** We claim that the numbers $a_1$ for which the sequence eventually becomes constant are exactly the $\boxed{\text{perfect squares}}$.

For the proof, suppose first that $a_1$ is a perfect square. Then $\lfloor \sqrt{a_1} \rfloor = \sqrt{a_1}$, so $a_2 = 2a_1 - \sqrt{a_1}^2 = a_1$, and by induction we get $a_n = a_1$ for all $n$, so the sequence is a constant sequence.

Conversely, suppose that the sequence is eventually constant, i.e., that there exists a positive integer $N$ such that $a_{n+1} = a_n$ for all $n \geq N$. Without loss of generality, we may assume that $N$ is the *smallest* such integer.

From the given recurrence, the condition $a_{n+1} = a_n$ implies $a_n = \lfloor \sqrt{a_n} \rfloor^2$, and thus that $a_n$ is a perfect square. Therefore $a_n$ is a perfect square for all $n \geq N$. If $N = 1$, then $a_1$ is a perfect square, and we are done.

Suppose now that $N > 1$. By the minimality assumption on $N$, we then have $a_N \neq a_{N-1}$. Applying the given recurrence again, we conclude that $\lfloor \sqrt{a_{N-1}} \rfloor \neq \sqrt{a_{N-1}}$, so $a_{N-1}$ is not a perfect square. Thus, setting $k = \lfloor \sqrt{a_{N-1}} \rfloor$, we have $a_{N-1} = k^2 + \ell$, where $1 \leq \ell \leq (k+1)^2 - k^2 - 1 = 2k$. But then $a_N = 2a_{N-1} - k^2 = k^2 + 2\ell \leq k^2 + 4k < (k+2)^2$. Hence $a_N$ must lie strictly between the two squares $k^2$ and $(k+2)^2$. Moreover, $a_N = k^2 + 2\ell$ cannot be equal to the square $(k+1)^2 = k^2 + 2k + 1$ for parity reasons. Therefore $a_N$ is not a perfect square, yielding a contradiction.

2. Is it possible to place 50 people in a circular room of radius 18 feet in a socially distant manner, i.e., such that no two of these people are less than 6 feet apart? Prove your answer.

**Solution.** The answer is $\boxed{\text{no}}$. For the proof, assume without loss of generality that the room is a disk $D$ of radius 18 centered at the origin, and suppose $P_1, \dots, P_n$ are points inside $D$ whose mutual distances are at least 6. We will show that $n$ must be strictly less than 50.

For $i = 1, \dots, n$ let $C_i$ be the disk of radius 3 centered at $P_i$. Our assumption that the distance between any two of the points $P_i$ is at least 6 implies that the disks $C_i$ are pairwise disjoint. Hence the area covered by these disks satisfies

$$A\left(\bigcup_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} A(C_i) = n\pi 3^2. \tag{1}$$

On the other hand, since the points $P_i$ lie inside the disk $D$ of radius 18 about the origin, the disks $C_i$ are all contained in a disk $D'$ of radius $18 + 3 = 21$ and centered at the origin. Therefore

$$A\left(\bigcup_{i=1}^{n} C_i\right) \leq A(D') = \pi 21^2 \tag{2}$$

Combining (1) and (2), we obtain $n\pi 3^2 \leq \pi 21^2$ and hence $n \leq 49$, which proves our claim.

3. Determine, with proof, whether or not there exists a permutation (i.e., re-ordering) $a_n$, $n = 1, 2, \dots$, of the natural numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}$$

converges.

**Solution.** We claim that such permutation does not exist; i.e., $\boxed{\text{the series } \sum_{n=1} a_n/n^2 \text{ diverges}}$.
To see this, consider partial sums of the form $S_N = \sum_{n=N}^{3N-1} a_n/n^2$. Since the sequence $\{a_n\}$ is a permutation of $1, 2, \dots$, at most $N$ of the $2N$ values $a_n$, $n = N, N+1, \dots, 3N-1$, can satisfy $a_n \le N$. Hence there exist at least $N$ such values with $a_n > N$. Therefore we have, for all $N \in \mathbb{N}$,

$$S_N > \frac{N}{(3N)^2} \#\{N \le n < 3N : a_n > N\} \ge \frac{N}{(3N)^2} \cdot N = \frac{1}{9},$$

and hence

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} = \sum_{i=0}^{\infty} S_{3^i} \ge \sum_{i=0}^{\infty} \frac{1}{9} = \infty,$$

as desired.

4. Prove that, for any real numbers $a_0, a_1, \dots, a_n$ and any **positive** real number $\alpha$,

$$\sum_{i,j=0}^{n} \frac{a_i a_j}{i+j+\alpha} \ge 0.$$

**Solution.** Let $p(x) = \sum_{i=0}^{n} a_i x^i$ and consider the integral $I = \int_0^1 x^{\alpha-1} p(x)^2\, dx$. Since $\alpha > 0$, the integral $I$ converges, and since the integrand is nonnegative, $I$ is nonnegative as well. On the other hand, expanding $p(x)^2$, we get

$$I = \int_0^1 x^{\alpha-1} \sum_{i,j=0}^{n} a_i a_j x^{i+j}\, dx = \sum_{i,j=0}^{n} a_i a_j \int_0^1 x^{i+j+\alpha-1}\, dx = \sum_{i,j=0}^{n} a_i a_j \frac{1}{i+j+\alpha}.$$

Thus the given sum is equal to $I$ and hence is nonnegative.

5. Given a set $A$ of numbers, let $S(A)$ denote the sum of the elements in $A$. (If $A$ is the empty set, we define $S(A) = 0$.) Determine, with proof, the set of all natural numbers $n$ for which there exists an integer $k$ such that the number of subsets $A \subset \{1, 2, \dots, n\}$ satisfying $S(A) \le k$ is equal to the number of subsets $A \subset \{1, 2, \dots, n\}$ satisfying $S(A) > k$.

**Solution.** We claim that the given property holds if and only if $\boxed{n \text{ is congruent to 1 or 2 modulo 4}}$.
For the proof, let $n \in \mathbb{N}$ be given, and for any subset $A \subset \{1, \dots, n\}$, let $A'$ denote the complement of $A$ in $\{1, \dots, n\}$. Observe that $S(A) + S(A') = \sum_{i=1}^{n} i = n(n+1)/2$. Thus, setting $s = n(n+1)/2$, we have $S(A) < s/2$ if and only if $S(A') > s/2$. It follows that the map $A \to A'$ is a bijection between the sets $A$ for which $S(A) < s/2$ and those for which $S(A) > s/2$. Therefore, the number of subsets $A$ with $S(A) < s/2$ is equal to the number of subsets $A$ with $S(A) > s/2$.
If $n \equiv 1, 2 \bmod 4$, then $s/2 = n(n+1)/4$ is not an integer, so a subset $A \subset \{1, \dots, n\}$ must satisfy either $S(A) < s/2$ or $S(A) > s/2$, and we see that the desired property holds with $k = \lfloor s/2 \rfloor$.
Now suppose $n \equiv 0, 3 \bmod 4$. In this case, $s/2$ is an integer. Let $k_0 = s/2$. We will show that there exists a set $A_0 \subset \{1, \dots, n\}$ with $S(A_0) = k_0$. Since there are $2^n$ subsets of $\{1, \dots, n\}$ and, by the above argument, the number of subsets $A$ with $S(A) \le k_0 - 1$ is equal to the number of

subsets $A$ with $S(A) \geq k_0+1$, it then follows that the number of subsets with $S(A) \leq k$ is strictly less than $2^n/2$ if $k \leq k_0 - 1$ and strictly greater than $2^n/2$ if $k \geq k_0$. Thus, the stated property cannot hold in the case $n \equiv 0, 3 \bmod 4$.

It remains to construct a set $A_0$ with $S(A_0) = k_0$. If $n \equiv 0 \bmod 4$, let $A_0 = \{1, 2, \dots, n/4\} \cup \{n, n-1, \dots, n+1-n/4\}$. Then $S(A_0) = (n+1)n/4 = k_0$. Similarly, if $n \equiv 3 \bmod 4$, then defining $A_0 = \{1, 2, \dots, (n+1)/4\} \cup \{n-1, n-2 \dots, n-(n+1)/4\}$ we have $S(A_0) = n(n+1)/4 = k_0$. This completes the proof.

6. Evaluate the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{1}{2^n}.$$

**Solution.** We will show that the series converges with sum $\boxed{1 - 1/\tan 1}$.

For the proof, set $S_n(x) = \sum_{k=1}^{n} \tan(x/2^k) 2^{-k}$. Then we need to show that $(*)$ $\lim_{n\to\infty} S_n(1) = 1 - 1/\tan 1$.

Let $P_n(x) = \prod_{k=1}^{n} \cos(x/2^k)$. Differentiating $\ln P_n(x)$ with respect to $x$, we obtain

$$(1) \qquad (\ln P_n(x))' = \sum_{k=1}^{n} \left(\ln \cos(x/2^k)\right)' = \sum_{k=1}^{n} \frac{-\sin(x/2^k)}{\cos(x/2^k)2^k} = -S_n(x).$$

On the other hand, using the identity $\cos(t) = \sin(2t)/(2 \sin t)$, we see that $P_n(x)$ is a telescoping product with value
$$P_n(x) = \prod_{k=1}^{n} \frac{\sin(x/2^{k-1})}{2\sin(x/2^k)} = \frac{\sin x}{2^n \sin(x/2^n)},$$
and therefore

$$(2) \qquad (\ln P_n(x))' = (\ln(\sin x) - \ln(2^n \sin(x/2^n)))' = \frac{\cos x}{\sin x} - \frac{\cos(x/2^n)}{2^n \sin(x/2^n)}.$$

Combining (1) and (2) and setting $x = 1$ yields
$$S_n(1) = -\frac{1}{\tan 1} + \frac{\cos(1/2^n)}{2^n \sin(1/2^n)}.$$

Since $\lim_{t\to 0} \cos t = 1$ and $\lim_{t\to 0} (\sin t)/t = 1$ as $t \to 0$, it follows that $\lim_{n\to\infty} S_n(1) = 1 - 1/\tan 1$, as claimed.

**Alternate solution:** We use a telescoping argument based on the double tangent formula

$$(3) \qquad \tan(2x) = \frac{2\tan x}{1 - (\tan x)^2} = \frac{-2}{\tan x - 1/\tan x}.$$

Let $t_k = \tan 1/2^k$, so that the given series is $\sum_{k=1}^{\infty} t_k/2^k$. Applying (3) with $x = 1/2^{k+1}$, we obtain
$$\begin{aligned} t_k &= \frac{-2}{t_{k+1} - 1/t_{k+1}}, \\ \frac{1}{2^k t_k} &= -\frac{t_{k+1}}{2^{k+1}} + \frac{1}{2^{k+1} t_{k+1}}, \\ \frac{t_{k+1}}{2^{k+1}} &= \frac{1}{2^{k+1} t_{k+1}} - \frac{1}{2^k t_k}. \end{aligned}$$

Hence
$$\sum_{k=0}^{n-1} \frac{t_{k+1}}{2^{k+1}} = \sum_{k=0}^{n-1} \left( \frac{1}{2^{k+1} t_{k+1}} - \frac{1}{2^k t_k} \right) = \frac{1}{2^n t_n} - \frac{1}{t_0} = \frac{\cos(1/2^n)}{2^n \sin(1/2^n)} - \frac{1}{\tan 1},$$
and letting $n \to \infty$, we obtain $\sum_{k=1}^{\infty} t_k/2^k = \sum_{k=0}^{\infty} t_{k+1}/2^{k+1} = 1 - 1/\tan 1$.