2021 UI UNDERGRADUATE MATH CONTEST PROBLEMS

1. Given a natural number a_1, define a sequence $\{a_n\}$ recursively by

$$a_{n+1} = 2a_n - \lfloor\sqrt{a_n}\rfloor^2 \quad (n = 1, 2, \dots),$$

where $\lfloor t \rfloor$ denotes the floor function, i.e., the greatest integer $\leq t$. For example, if $a_1 = 6$, then $a_2 = 2 \cdot 6 - 2^2 = 8$, $a_3 = 2 \cdot 8 - 2^2 = 12$, $a_4 = 2 \cdot 12 - 3^2 = 15$, and so on. Determine, with proof, all natural numbers a_1 for which the sequence $\{a_n\}$ eventually becomes constant.

2. Is it possible to place 50 people in a circular room of radius 18 feet in a socially distant manner, i.e., such that no two of these people are less than 6 feet apart? Prove your answer.

3. Determine, with proof, whether or not there exists a permutation (i.e., re-ordering) a_n, $n = 1, 2, \dots$, of the natural numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}$$

converges.

4. Prove that, for any real numbers $a_0, a_1, \dots, a_n$ and any **positive** real number α,

$$\sum_{i,j=0}^{n} \frac{a_i a_j}{i + j + \alpha} \geq 0.$$

5. Given a set A of numbers, let $S(A)$ denote the sum of the elements in A. (If A is the empty set, we define $S(A) = 0$.) Determine, with proof, the set of all natural numbers n for which there exists an integer k such that the number of subsets $A \subset \{1, 2, \dots, n\}$ satisfying $S(A) \leq k$ is equal to the number of subsets $A \subset \{1, 2, \dots, n\}$ satisfying $S(A) > k$.

6. Evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{1}{2^n}.$$