2020 UI UNDERGRADUATE MATH CONTEST
Solutions

1. Let $x_1, x_2, \ldots, x_{2020}$ be the roots of the polynomial $P(x) = x^{2020} + 2020x - 1$. Find, with proof, the sum

$$\sum_{i=1}^{2020} x_i^{2020}.$$

Solution. We will show that the given sum is $\boxed{2020}$.
Since each x_i is a root of $P(x)$, we have $x_i^{2020} + 2020x_i - 1 = 0$ and therefore $x_i^{2020} = -2020x_i + 1$ for all i. Hence,

$$\sum_{i=1}^{2020} x_i^{2020} = -2020S + 2020,$$

where $S = \sum_{i=1}^{2020} x_i$ is the sum of the roots of $P(x)$. By the factorization theorem, we have

$$P(x) = \prod_{i=1}^{2020}(x - x_i) = x^{2020} - x^{2019}S + \cdots,$$

where the dots represent terms of order x^{2018} or lower. Since the given polynomial has no term involving x^{2019}, we must have $S = 0$. Hence the given sum is 2020.

2. Without calculating the expressions involved, show that

$$2019^{2018+2020} < 2018^{2018}2020^{2020}.$$

Solution. Let $x = 2018$, $y = 2020$ and $z = (x + y)/2 = 2019$. The inequality to be proved then becomes

$$z^{2z} < x^x y^y. \tag{1}$$

Taking logarithms on both sides, we see that (1) holds if and only

$$z \log z < \frac{1}{2}(x \log x + y \log y). \tag{2}$$

Let $f(t) = t \log t$. Then $f'(t) = \log t + 1$ and $f''(t) = 1/t > 0$ for $t > 0$, so $f(t)$ is a convex function on $(0, \infty)$. Hence

$$f(z) = f\left(\frac{x + y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)). \tag{3}$$

Since $f''(t) > 0$ for all $t > 0$, f cannot be linear on the interval $[x, y]$, so the inequality in (3) must be strict. This proves (2).

3. Call a point (x, y) in the plane *hyperbolic* if it lies on one of the hyperbolas $y = 1/x$ or $y = -1/x$. Find a square such that its four vertices and the midpoints of its four sides are all hyperbolic.

Solution. We will show that the square centered at the origin whose midpoints of the sides are given by $\pm(\sqrt{\Phi}, 1/\sqrt{\Phi})$ and $\pm(1/\sqrt{\Phi}, -\sqrt{\Phi})$, where $\Phi = (\sqrt{5}+1)/2$, has this property.

More generally, let $\pm\vec{u}, \pm\vec{v}$ denote the vectors representing the midpoints of the four sides of a square centered at the origin. Then the four vertices are given by the vectors $\pm\vec{u} \pm \vec{v}$. These vertices form a square if and only if the vectors $\vec{u}$ and $\vec{v}$ are orthogonal and of the same length. Thus, we must construct $\vec{u}$ and $\vec{v}$ satisfying this "square condition" and such that the points $\pm\vec{u}$, $\pm\vec{v}$, and $\pm\vec{u} \pm \vec{v}$ are all hyperbolic.

Taking $\vec{u}$ and $\vec{v}$ of the form $\vec{u} = (t, 1/t)$ and $\vec{v} = (-1/t, t)$, $t > 0$, ensures that the points $\pm\vec{u}$ and $\pm\vec{v}$ are hyperbolic and that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal and of the same length.

It remains to ensure that that the four points representing the vertices, $\pm\vec{u}\pm\vec{v}$ are also hyperbolic. These points are of the form $\pm(t-1/t, t+1/t)$ or $\pm(t+1/t, -t+1/t)$, so the hyperbolicity requirement is equivalent to

$$\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right) = \pm 1. \tag{1}$$

This equation can be written as $t^2 - 1/t^2 = \pm 1$, which is a quadratic equation in $x = t^2$. Taking the plus sign we obtain $x^2 - x = 1$, which has the Golden Ratio $\Phi = (\sqrt{5}+1)/2$ as one of its two solutions. Hence, $t = \sqrt{\Phi}$ is a particular solution to (1). Therefore, the square with midpoints given by $\pm(\sqrt{\Phi}, 1/\sqrt{\Phi})$ and $\pm(1/\sqrt{\Phi}, -\sqrt{\Phi})$ has the desired properties.

Remarks: The other solutions to (1) lead to rotated or reflected versions of the above square. The vertices of the square constructed above are $\pm(\sqrt{\sqrt{5}+2}, \sqrt{\sqrt{5}-2})$ and $\pm(\sqrt{\sqrt{5}-2}, \sqrt{\sqrt{5}+2})$.

4. Let $n \geq 3$ and let S be a family of 2^{n-1} non-empty subsets of $\{1, 2, \ldots, n\}$ such that any three sets in S have a non-empty intersection. Show that there exists an element in $\{1, 2, \ldots, n\}$ that belongs to every set in S.

Solution. We first observe that, for any set $A \subset \{1, \ldots, n\}$, exactly one of the sets A and A^c must be in S. Indeed, since A and A^c are disjoint, S can contain at most one of these sets. On the other hand, there are 2^{n-1} unordered pairs $\{A, A^c\}$, $A \subset \{1, \ldots, n\}$, and there are also 2^{n-1} sets in S. Therefore S must contain exactly one element from each unordered pair $\{A, A^c\}$.

We now claim that if $A_1, \ldots, A_k$ are all in S, then the set $A_1 \cap \cdots \cap A_k$ is also in S. The result follows from this claim by taking $A_1, \ldots, A_k$ to be *all* 2^{n-1} sets in S. Since, by assumption, S contains only non-empty sets, it follows that the intersection of all sets in S must be non-empty as well.

To prove the claim, we proceed by induction on k. The case $k = 1$ is trivial, so let $k \geq 2$ be given and assume that the claim holds for $k' = k - 1$. Let $A_1, \ldots, A_k$ be k sets in S, and let $B_k = A_1 \cap \cdots \cap A_k$. Note that $B_k = A_k \cap B_{k-1}$, where $B_{k-1} = A_1 \cap \cdots \cap A_{k-1}$. By the induction hypothesis, the set B_{k-1} is in S. By the above observation, either $A_k \cap B_{k-1}$ or $(A_k \cap B_{k-1})^c$ must be in S. If $A_k \cap B_{k-1}$ is in S, then $A_1 \cap \cdots \cap A_k$ is in S and the induction step is complete. On the other hand, if $(A_k \cap B_{k-1})^c$ is in S, then the assumption that any three sets in S have a non-empty intersection would be violated for sets A_k, B_{k-1}, and $(A_1 \cap B_{k-1})^c$. Thus, this case cannot occur. This completes the proof.

Remark: In fact, the intersection of all sets in S consists of *exactly* one element a, for otherwise S would have fewer than 2^{n-1} elements. Moreover, since there are exactly 2^{n-1} subsets of $\{1, 2, \ldots, n\}$ containing a, S must consist of *all* of these subsets.

5. Find all nonzero polynomials $P(x)$ with nonnegative integer coefficients such that $P(n)$ divides n^{2020} for infinitely many positive integers n.

Solution. We claim that the polynomials with the given property are $P_0(x) = x^{2020}$ and $P_{a,k}(x) = ax^k$, where $a \in \mathbb{N}$ and $k \in \{1, 2, \ldots, 2019\}$.

Sufficiency. We first show that the above polynomials have the desired divisibility property. Obviously, $P_0(n) = n^{2020}$ divides n^{2020} for all $n \in \mathbb{N}$. Moreover, given $a \in \mathbb{N}$ and $k \in \{1, 2, \ldots, 2019\}$, we have $n^{2020}/P_{a,k}(n) = n^{2020-k}/a$, which is an integer whenever n is a multiple of a. Thus, $P_{a,k}(n)$ divides n^{2020} for infinitely many n.

Necessity. It remains to show that $P(x)$ must necessarily be of the above form. By the Division Algorithm, we have $x^{2020} = P(x)q(x) + r(x)$, where q and r are polynomials with *rational* coefficients and $\deg r < \deg P$. Multiplying by the least common multiple of the denominators of the coefficients of these polynomials, we obtain $bx^{2020} = P(x)Q(x) + R(x)$, where b is a positive integer and $Q(x)$ and $R(x)$ are polynomials with *integer* coefficients such that $\deg R < \deg P$. Dividing this relation by $P(x)$ and restricting x to integer values n gives

$$\frac{bn^{2020}}{P(n)} - Q(n) = \frac{R(n)}{P(n)}. \tag{1}$$

Let S denote the set of positive integers n for which $P(n)$ divides n^{2020}. Then $bn^{2020}/P(n)$ is an integer for all $n \in S$, and since $Q(n)$ is also an integer (since $Q(x)$ is a polynomial with integer coefficients), it follows that the left side of (1) is an integer for all $n \in S$. Hence $R(n)/P(n)$ must be an integer for all $n \in S$.

Now recall that, since $\deg R < \deg P$, we have $\lim_{x\to\infty} R(x)/P(x) = 0$, so there exists some $N \geq 0$ such that $|R(n)/P(n)| < 1$ for all $n \geq N$. But since $R(n)/P(n)$ is an integer for $n \in S$, this implies $R(n)/P(n) = 0$, and hence $R(n) = 0$, for all $n \in S$ with $n \geq N$. Since S is infinite, it follows that there are infinitely many n with $R(n) = 0$, so the polynomial $R(x)$ must be the zero polynomial.

Therefore we have the polynomial relation $bx^{2020} = P(x)Q(x)$. Since P and Q are polynomials with integer coefficients, this can only hold if $P(x) = ax^k$ and $Q(x) = (b/a)x^{2020-k}$ for some $k \in \{0, 1, \ldots, 2020\}$ and $a|b$. In particular, P is necessarily of the form $P(x) = ax^k$ for some $a \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2020\}$. Moreover, when $k = 2020$, a direct check of the divisibility condition $P(n)|n^{2020}$ shows that this condition can only hold when $a = 1$. This completes the proof.

6. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(f(n)) = n^2$ for all $n \in \mathbb{N}$? Justify your answer.

Solution. We will show that such a function does exist.

Set $f(1) = 1$ (so that the desired relation $f(f(n)) = n^2$ holds for $n = 1$) and for integers $n \geq 2$ define f as follows:

Let $n_1, n_2, \ldots$ be an enumeration of the natural numbers ≥ 2 which are not complete squares. Then any integer $n \geq 2$ has a unique representation of the form $n = n_k^{2^m}$, with $k \in \mathbb{N}$ and $m \in \{0, 1, \ldots\}$. Define f by

$$f(n_k^{2^m}) = \begin{cases} n_{k+1}^{2^m} & \text{if } k \text{ is odd,} \\ n_{k-1}^{2^{m+1}} & \text{if } k \text{ is even.} \end{cases}$$

Then

$$f(f(n_k^{2^m})) = \begin{cases} f(n_{k+1}^{2^m}) = n_k^{2^{m+1}} & \text{if } k \text{ is odd,} \\ f(n_{k-1}^{2^{m+1}}) = n_k^{2^{m+1}} & \text{if } k \text{ is even,} \end{cases}$$

Thus, f satisfies $f(f(n)) = n^2$ for all $n \in \mathbb{N}$.

7. Let P_n be the probability that a random point (x, y), selected uniformly from the unit square $[0, 1] \times [0, 1]$, satisfies $x^n + y^n > 1$.

   (a) Show that the limit $L = \lim_{n\to\infty} n^2 P_n$ exists and satisfies $0 < L < \infty$.

   (b) Evaluate L.

Solution. We claim that $\lim_{n\to\infty} n^2 P_n = \pi^2/6$.

The probability P_n is the area of the part of the unit square that lies above the curve $x^n + y^n = 1$, or equivalently, above the graph of the function $y = (1 - x^n)^{1/n}$. Thus,

$$\begin{aligned} n^2 P_n &= n^2 \int_0^1 \left(1 - (1 - x^n)^{1/n}\right) dx \\ &= n^2 \int_0^1 \frac{1 - (1-u)^{1/n}}{n u^{1-1/n}}\, du \quad (\text{set } x = u^{1/n},\ dx = (1/n)u^{1/n-1} du) \\ &= \int_0^1 \frac{n\left(1 - (1-u)^{1/n}\right)}{u} \cdot u^{1/n}\, du =: I_n. \end{aligned} \tag{1}$$

Thus, it suffices to show that $\lim_{n\to\infty} I_n = \pi^2/6$. We will do so by showing the following relations:

$$\lim_{n\to\infty} I_n = \int_0^1 \frac{-\log(1-u)}{u} du, \tag{2}$$

$$\int_0^1 \frac{-\log(1-u)}{u} du = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{3}$$

Proof of (3): Using the power series $-\log(1-u) = \sum_{k=1}^{\infty} u^k/k$ and the Monotone Convergence Theorem (noting that every term in the summand is nonnegative) we have

$$\int_0^1 \frac{-\log(1-u)}{u}\, du = \int_0^1 \frac{1}{u} \sum_{k=1}^{\infty} \frac{u^k}{k}\, du \overset{\text{MCT}}{=} \sum_{k=1}^{\infty} \int_0^1 \frac{u^{k-1}}{k}\, du = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

The latter series is equal to $\pi^2/6$ by a famous result of Euler. This proves (3).

Proof of (2): By L'Hôpital's rule, we have, for each $u \in (0,1)$,

$$\begin{aligned} \lim_{n\to\infty} n(1 - (1-u)^{1/n}) u^{1/n} &= \lim_{x\to 0} \frac{1 - (1-u)^x}{x} u^x \\ &= \lim_{x\to 0} \frac{1 - (1-u)^x}{x} \overset{\text{LH}}{=} -\log(1-u). \end{aligned} \tag{4}$$

Since

$$1 - (1-u)^{1/n} = 1 - e^{\frac{1}{n}\ln(1-u)} \le -\frac{1}{n} \ln(1-u)$$

for all $u \in (0,1)$, the integrand in (1) is *bounded above* by $(-\log(1-u))/u$ for all $u \in (0,1)$ and all n. Moreover, as shown above, the integral $\int_0^1 \frac{-\ln(1-u)}{u} du$ converges (absolutely). Hence by the Dominated Convergence Theorem we can take the limit as $n \to \infty$ in (1) inside the integral, and using (4) we obtain

$$\lim_{n\to\infty} \int_0^1 \frac{n\left(1 - (1-u)^{1/n}\right)}{u} \cdot u^{1/n}\, du = \int_0^1 \frac{-\log(1-u)}{u}\, du.$$

This proves (2).

Alternative approach: An alternative approach, suggested by Jun Loo, is to use the binomial series expansion $(1+t)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} t^k$. Applying this expansion with $\alpha = 1/n$ and $t = -x^n$, we get

$$(1 - x^n)^{1/n} = \sum_{k=0}^{\infty} \binom{1/n}{k} (-1)^k x^{kn},$$

and hence

$$
\text{(5)} \qquad \begin{aligned}
n^2 P_n &= n^2 \int_0^1 \left(1 - (1 - x^n)^{1/n}\right) dx \\
&= -\int_0^1 n^2 \sum_{k=1}^{\infty} \binom{1/n}{k} (-1)^k x^{kn} dx \\
&= -\sum_{k=1}^{\infty} n^2 \binom{1/n}{k} (-1)^k \int_0^1 x^{kn} dx \\
&= \sum_{k=1}^{\infty} n^2 \binom{1/n}{k} \frac{(-1)^{k-1}}{kn+1},
\end{aligned}
$$

though the interchange of summation and integration in the third step requires some justification. Now note that, for each *fixed* $k \geq 1$,

$$
\begin{aligned}
\lim_{n\to\infty} n^2 \binom{1/n}{k} \frac{1}{kn+1} &= \lim_{n\to\infty} n^2 \frac{\left(\frac{1}{n}\right)\left(-1+\frac{1}{n}\right)\left(-2+\frac{1}{n}\right)\ldots\left(-k+1+\frac{1}{n}\right)}{1\cdot 2\cdots k} \cdot \frac{1}{kn+1} \\
&= \lim_{n\to\infty} \frac{\left(-1+\frac{1}{n}\right)\left(-2+\frac{1}{n}\right)\ldots\left(-k+1+\frac{1}{n}\right)}{1\cdot 2\cdots k} \cdot \frac{1}{k+\frac{1}{n}} = \frac{(-1)^{k-1}}{k^2}.
\end{aligned}
$$

Hence, if we take the termwise limit as $n \to \infty$ in the final series of (5) we obtain the series $\sum_{k=1}^{\infty} 1/k^2$, and thus get the same answer as before. However, as before an additional argument is needed to justify the interchanging of limits and summation.